

REDUCTION OF A SYMPLECTIC-LIKE LIE ALGEBROID WITH MOMENTUM MAP AND ITS APPLICATION TO FIBERWISE LINEAR POISSON STRUCTURES

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ABSTRACT. This article addresses the problem of developing an extension of the Marsden-Weinstein reduction process to symplectic-like Lie algebroids, and in particular to the case of the canonical cover of a fiberwise linear Poisson structure, whose reduction process is the analogue to cotangent bundle reduction in the context of Lie algebroids.

Dedicated to the memory of Jerrold E. Marsden

1. INTRODUCTION

1.1. Preliminaries. A smooth and proper action of a Lie group G on a symplectic manifold (M, Ω) is called Hamiltonian if G acts by symplectomorphisms and it admits a coadjoint equivariant momentum map $J : M \rightarrow \mathfrak{g}^*$ satisfying the compatibility condition

$$\Omega(\xi_M, \cdot) = d\langle J(\cdot), \xi \rangle \quad \forall \xi \in \mathfrak{g},$$

where $\xi_M \in \mathfrak{X}(M)$ is the fundamental vector field corresponding to the Lie algebra element ξ . The Marsden-Weinstein symplectic reduction process, introduced in [20] states that, if μ is a regular value of J , and G_μ , the stabilizer of μ for the coadjoint representation, acts freely and properly on $J^{-1}(\mu)$, then the quotient $J^{-1}(\mu)/G_\mu$ is a smooth manifold with a naturally induced “reduced” symplectic form Ω_μ . The identity characterizing Ω_μ is

$$\iota_\mu^* \Omega = \pi_\mu^* \Omega_\mu,$$

where $\iota_\mu : J^{-1}(\mu) \hookrightarrow M$ and $\pi_\mu : J^{-1}(\mu) \rightarrow J^{-1}(\mu)/G_\mu$ are the natural inclusion and projection, respectively.

One can look at the particular and important case when our symplectic structure is not on M , but on its cotangent bundle T^*M , and the action of G is the cotangent lift of an action on M . In this case, the lifted action is automatically Hamiltonian with respect to the canonical

Key words and phrases. Reduction, Lie algebroids, symplectic-like Lie algebroids, momentum maps.

Mathematics Subject Classification (2010): 53D17, 53D20, 37J15, 53D05.

The authors have been partially supported by MEC (Spain) grants MTM2009-13383 and MTM2009-08166-E. The research of M.R-O has been also partially supported by a Marie Curie Intra European Fellowship PIEF-GA-2008-220239 and a Marie Curie Reintegration Grant PERG-GA-2010-27697. The research of J.C.-M and E-P has been also partially supported by the grants of the Canary government SOLSUBC200801000238 and ProID20100210. We also would like to thank D. Iglesias, D. Martín de Diego and E. Martínez for their useful comments.

symplectic form on T^*M given in trivializing local coordinates by $\Omega_c = dx^i \wedge dy^i$. It can be shown that an equivariant momentum map for this action is given by

$$(1.1) \quad J(\alpha_x)(\xi) = \alpha_x(\xi_M(x)) \quad \text{for all } \alpha_x \in T_x^*M, \xi \in \mathfrak{g}.$$

The reduction theory for lifted actions on cotangent bundles was first studied in [26], where only the case of Abelian actions was addressed. The general case was treated in [12] and [1]. The set of results emerging from those and other references is usually known as cotangent bundle reduction. We will expose here the basic lines of this subject and refer to [18, 23] for a more detailed survey. We will assume from now on that the action of G on M is free and proper.

For the case of lifted actions, due to the particularities of the fibered geometry existent, we can distinguish different situations for the choice of momentum value. This cases are 1) $\mu = 0$, 2) $G_\mu = G$ and 3) general values of J . The theory of cotangent bundle reduction establishes the existence of maps from the abstract symplectic reduced spaces to certain cotangent bundles equipped with canonical symplectic forms possibly deformed by a *magnetic term*. The different possibilities are:

- $\mu = 0$. There is a symplectomorphism

$$\phi_0 : (J^{-1}(0)/G, \Omega_0) \rightarrow (T^*(M/G), \Omega_c).$$

- $G_\mu = G$. There is a symplectomorphism

$$\phi_\mu : (J^{-1}(\mu)/G_\mu, \Omega_\mu) \rightarrow (T^*(M/G_\mu), \Omega_c - B_\mu).$$

- General μ . There is a symplectic embedding

$$\phi_\mu : (J^{-1}(\mu)/G_\mu, \Omega_\mu) \rightarrow (T^*(M/G_\mu), \Omega_c - B_\mu).$$

In the last two cases the *magnetic term* B_μ is the pullback by the cotangent bundle projection of a closed two-form on M/G_μ . This two-form is obtained, for example, via the choice of a principal connection for the fibration $M \rightarrow M/G_\mu$.

Note also that in the case $G_\mu = G$ (which corresponds to values of J for which their coadjoint orbits are trivial) we have $M/G_\mu = M/G$. Therefore, topologically, all the reduced spaces for momentum values μ with trivial coadjoint orbits are equivalent to the same space $T^*(M/G)$, and their symplectic forms differ only possibly in the terms B_μ .

This paper develops a generalization of the reduction theory reviewed above for general symplectic manifolds and the particular case of cotangent bundles, to the setup of Lie algebroids. For general Marsden-Weinstein reduction, this happens when one substitutes the tangent bundle TM of a symplectic manifold M by a more general symplectic vector bundle A over M (a symplectic-like Lie algebroid). For cotangent bundle reduction, the generalization consists in substituting TM by a general Lie algebroid A , and $T(T^*M)$ by a special construction called the canonical cover (or the prolongation of A over A^* following the terminology of [13]) of A^* , which happens to be a symplectic-like Lie algebroid. The latter generalization is a particular case of the former, and both cases coincide with Marsden-Weinstein reduction and cotangent bundle reduction, respectively, when A is just the tangent bundle of M . In the remainder of this section we will give an overview of the new results of this article.

1.2. Reduction for Lie algebroids. A Lie algebroid is a natural generalization of the tangent bundle to a manifold. It consists of a vector bundle $A \rightarrow M$ equipped with a certain geometric structure that allows to generalize on the one hand, the Lie algebra of vector fields on M to a Lie algebra structure $[\cdot, \cdot]$ on the space of sections of A , and on the other, the exterior derivative on differential forms to the a derivation d^A of the exterior algebra of multi-sections of A^* . The general theory of Lie algebroids is reviewed in Section 2. We remark that giving a Lie algebroid structure on vector bundle A is equivalent to giving a linear Poisson bivector on the dual vector bundle A^* of A .

In order to study the reduction process for a Lie algebroid $A \rightarrow M$ we introduce in Subsection 3.1 the notion of an action by complete lifts on A as an action $\Phi : G \times A \rightarrow A$ of a Lie group G by vector bundle automorphisms of a Lie group G on A together with a Lie algebra anti-morphism $\psi : \mathfrak{g} \rightarrow \Gamma(A)$ such that the infinitesimal generator of $\xi \in \mathfrak{g}$ with respect to Φ is just the complete lift of $\psi(\xi)$; or equivalently, an action $\Phi : G \times A^* \rightarrow A^*$ on the dual vector bundle A^* by Poisson automorphisms such that the infinitesimal generator of ξ is just the Hamiltonian vector field (with respect to the linear Poisson structure on A^*) of the linear function associated with the section $\psi(\xi) \in \Gamma(A)$. The standard example of an action by complete lifts on the Lie algebroid TM is the tangent lift of an action on M .

If $\Phi : G \times A \rightarrow A$ is a free and proper action of a connected Lie group G on the Lie algebroid A by complete lifts then in Section 3 we construct an affine action $\Phi^T : TG \times A \rightarrow A$ of the tangent Lie group TG such that the orbit space A/TG is a Lie algebroid over the reduced manifold M/G corresponding to the induced action $\phi : G \times M \rightarrow M$ of G on the base manifold M of the Lie algebroid A . Moreover, we prove that the projection $\tilde{\pi} : A \rightarrow A/TG$ is a Lie algebroid morphism (see Theorem 3.6).

1.3. Reduction for symplectic-like Lie algebroids. The main idea behind the generalization of symplectic reduction to Lie algebroids consists in realizing that a symplectic manifold can be seen as a Lie algebroid endowed with a symplectic vector space structure on each fiber varying smoothly. Under this point of view, the Lie algebroid is nothing but the tangent bundle of the symplectic manifold, and the symplectic structure on the fibers is the evaluation of the symplectic form to each point. The fact that the symplectic form is closed can then be interpreted as being closed as a differential two-form on the Lie algebroid. This situation can be extended to an arbitrary Lie algebroid, not necessarily the tangent bundle of a symplectic manifold. Therefore, the setup for this paper will be a symplectic-like Lie algebroid, i.e. a Lie algebroid $A \rightarrow M$ equipped with a non-degenerate smooth 2-section $\Omega \in \Gamma(\wedge^2 A^*)$ satisfying $d^A \Omega = 0$ and an action $\Phi : G \times A \rightarrow A$ of a Lie group G by complete lifts on A .

The main result of Subsection 3.2 is to obtain a Lie algebroid version of the Marsden-Weinstein reduction for symplectic manifolds. Firstly, we will consider a momentum map $J : M \rightarrow \mathfrak{g}$ for the action $\phi : G \times M \rightarrow M$ which allows to define an equivariant map $J^T : A \rightarrow \mathfrak{g}^* \times \mathfrak{g}^*$ for the affine action $\Phi^T : TG \times A \rightarrow A$. Then, in Theorem 3.11, we describe the Lie algebroid analogue of the Marsden-Weinstein reduction scheme. It states that under a regularity condition involving a value $\mu \in \mathfrak{g}^*$ of J , the quotient $A_\mu := (J^T)^{-1}(0, \mu)/TG_\mu$ is a symplectic-like Lie algebroid over $J^{-1}(\mu)/G_\mu$. If Ω is the symplectic-like section on A and $\tilde{\pi}_\mu : (J^T)^{-1}(0, \mu) \rightarrow A_\mu$ and $\tilde{\iota}_\mu : (J^T)^{-1}(0, \mu) \rightarrow A$ are the canonical projection and inclusion respectively, then the

reduced symplectic-like section Ω_μ on A_μ is characterized by the condition

$$\tilde{\pi}_\mu^* \Omega_\mu = \tilde{\iota}_\mu^* \Omega.$$

It is well-known that the base manifold of a symplectic-like Lie algebroid has an induced Poisson structure (see [13, 11, 16]). Then, as a consequence of the reduction theorem for symplectic-like Lie algebroids, it is shown in Theorem 3.13 that the Poisson structures on the base manifolds of the original and reduced symplectic-like Lie algebroids are related in a similar way. Namely, if $\{\cdot, \cdot\}$ denotes the Poisson structure on M induced by Ω and $\{\cdot, \cdot\}_\mu$ is the corresponding structure on $J^{-1}(\mu)/G_\mu$ induced by the reduced symplectic-like section Ω_μ , then

$$\{\tilde{f}, \tilde{g}\}_\mu \circ \pi_\mu = \{f, g\} \circ i_\mu,$$

where $\pi_\mu : J^{-1}(\mu) \rightarrow J^{-1}(\mu)/G_\mu$ and $i_\mu : J^{-1}(\mu) \rightarrow M$ are the canonical projection and the inclusion, respectively, \tilde{f}, \tilde{g} are functions on $J^{-1}(\mu)/G_\mu$ and f, g are G -invariant extensions to M of $\tilde{f} \circ \pi_\mu$ and $\tilde{g} \circ \pi_\mu$, respectively. That is, the reduction obtained on the base manifold of the Lie algebroid is just the Marsden-Ratiu reduction for Poisson manifolds [19].

In [2] a theory of reduction for Courant algebroids is presented. A symplectic-like Lie algebroid A induces a Lie bialgebroid and therefore a Courant algebroid on $A \oplus A^*$ (see [14]). Then, one may apply this Courant reduction process to $A \oplus A^*$ and could recover, after a long computation, some results described in Section 3.2. However, we focus our study in the reduction of the particular case of symplectic-like Lie algebroids which allows us to obtain more explicit results on this type of reduction.

1.4. Reduction for canonical covers of fiberwise linear Poisson structures. Section 4 studies, within the framework of symplectic-like Lie algebroids, the situation equivalent to cotangent bundle reduction. In this case the generalization goes as follows: First, the cotangent bundle over a manifold M is replaced by A^* , the dual of a Lie algebroid $A \rightarrow M$, and then we consider the canonical cover of A^* , (also known as the prolongation of A over A^*), denoted by $\mathcal{T}^A A^*$. This is a natural construction on the dual of a Lie algebroid, which happens to be in a canonical way, a symplectic-like Lie algebroid with base manifold A^* . If A is the tangent bundle of M , then $\mathcal{T}^A A^*$ is just $T(T^*M)$. If there is a suitable action of a Lie group G by complete lifts on A , this action can be further lifted to the canonical cover of A^* , in a natural way, and this lifted action happens to be a morphism of symplectic-like Lie algebroids. Furthermore, one may define an equivariant momentum map on A^* (the base space of $\mathcal{T}^A A^*$) in a similar way when as how the classical momentum map (1.1) on T^*M is introduced. In general it is not possible to find an equivariant momentum map for a Poisson action (see, for instance, [8]). However, for the case of the Poisson action $\Phi^* : G \times A^* \rightarrow A^*$ associated with an action $\Phi : G \times A \rightarrow A$ by complete lifts, an equivariant momentum map is described.

Applying the reduction theory of symplectic-like Lie algebroids just developed we know that the reduction of $\mathcal{T}^A A^*$ at any momentum value is again a symplectic-like Lie algebroid. However, as in the situation of cotangent bundle reduction it is expected that the extra properties of the symplectic-like Lie algebroid, in this case the prolonged fibered structure, will be recovered in the quotient in some way. This is the content of the results of Section 5, for which the obtained new results reduce to the standard cotangent bundle reduction theory in the case that the starting Lie algebroid $A \rightarrow M$ is the standard Lie algebroid TM . In Subsection 5.1 it is shown

(Theorem 5.1) that if $\mu = 0$, there is a symplectic-like Lie algebroid isomorphism between the reduced symplectic-like Lie algebroid $(J^T)^{-1}(0, 0)/TG$ and $\mathcal{T}^{A_0}A_0^*$. Here $A_0 \rightarrow M/G$ is a Lie algebroid with total space A/TG . The case $G_\mu = G$ is studied in Subsection 5.2. There it is shown that $(J^T)^{-1}(0, \mu)/TG_\mu$ is also isomorphic to $\mathcal{T}^{A_0}A_0^*$, but in this case this isomorphism is canonical between the symplectic-like Lie algebroids if the canonical symplectic-like section on $\mathcal{T}^{A_0}A_0^*$ is modified by the addition of a twisting term which consists in the lift to $\mathcal{T}^{A_0}A_0^*$ of a closed 2-section of A_0^* . This is the content of Theorem 5.2. Finally, Subsection 5.3, in its main result, Theorem 5.3 shows that for the most general momentum values, $(J^T)^{-1}(0, \mu)/TG_\mu$ is canonically embedded as a Lie subalgebroid of $\mathcal{T}^{A_{0,\mu}}A_{0,\mu}^*$, where $A_{0,\mu}$ is a Lie algebroid A/TG_μ over M/G_μ , and the prologation $\mathcal{T}^{A_{0,\mu}}A_{0,\mu}^*$ is equipped with its canonical symplectic-like section minus a magnetic term, just as in the $G_\mu = G$ case.

As far as we know there is a similar research being done independently by E. Martínez [22]. In addition, in the same direction, some similar results in the more general setting of Lie bialgebroids has been discussed in [25].

2. LIE ALGEBROIDS

Let A be a vector bundle of rank n over the manifold M of dimension m and let $\tau : A \rightarrow M$ be its vector bundle projection. Denote by $\Gamma(A)$ the $C^\infty(M)$ -module of sections of $\tau : A \rightarrow M$. A *Lie algebroid structure* $(\llbracket \cdot, \cdot \rrbracket, \rho)$ on A is a Lie bracket $\llbracket \cdot, \cdot \rrbracket$ on the space $\Gamma(A)$ and a bundle map $\rho : A \rightarrow TM$, called *the anchor map*, such that, if we also denote by $\rho : \Gamma(A) \rightarrow \mathfrak{X}(M)$ the homomorphism of $C^\infty(M)$ -modules induced by the anchor map satisfying

$$(2.2) \quad \llbracket X, fY \rrbracket = f\llbracket X, Y \rrbracket + \rho(X)(f)Y, \text{ for } X, Y \in \Gamma(A) \text{ and } f \in C^\infty(M).$$

The triple $(A, \llbracket \cdot, \cdot \rrbracket, \rho)$ is called a *Lie algebroid over M* (see [15]). In such a case, the anchor map $\rho : \Gamma(A) \rightarrow \mathfrak{X}(M)$ is a homomorphism between the Lie algebras $(\Gamma(A), \llbracket \cdot, \cdot \rrbracket)$ and $(\mathfrak{X}(M), [\cdot, \cdot])$. If $(A, \llbracket \cdot, \cdot \rrbracket, \rho)$ is a Lie algebroid, one can define a cohomology operator, which is called *the differential of A* , $d^A : \Gamma(\wedge^k A^*) \rightarrow \Gamma(\wedge^{k+1} A^*)$, as follows

$$(2.3) \quad \begin{aligned} (d^A \mu)(X_0, \dots, X_k) &= \sum_{i=0}^k (-1)^i \rho(X_i)(\mu(X_0, \dots, \widehat{X}_i, \dots, X_k)) \\ &+ \sum_{i < j} (-1)^{i+j} \mu(\llbracket X_i, X_j \rrbracket, X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k), \end{aligned}$$

for $\mu \in \Gamma(\wedge^k A^*)$ and $X_0, \dots, X_k \in \Gamma(A)$. Moreover, if $X \in \Gamma(A)$ one may introduce, in a natural way, *the Lie derivate for multisections of A^* with respect to X* , as the operator $\mathcal{L}_X^A : \Gamma(\wedge^k A^*) \rightarrow \Gamma(\wedge^k A^*)$ given by $\mathcal{L}_X^A = i_X \circ d^A + d^A \circ i_X$ (see [15]). *The Lie derivative of a multisection $P \in \Gamma(\wedge^k A)$ of A with respect to X* is the k -section $\mathcal{L}_X^A P$ on A characterized by

$$\mathcal{L}_X^A P(\alpha_1, \dots, \alpha_k) = \rho(X)(P(\alpha_1, \dots, \alpha_k)) - \sum_i P(\alpha_1, \dots, \mathcal{L}_X^A \alpha_i, \dots, \alpha_k)$$

with $\alpha_i \in \Gamma(A^*)$.

If $(A, \llbracket \cdot, \cdot \rrbracket, \rho)$ is a Lie algebroid, we have a natural linear Poisson structure Π_{A^*} on the dual vector bundle A^* characterized as follows:

$$(2.4) \quad \begin{aligned} \{\widehat{X}, \widehat{Y}\}_{\Pi_{A^*}} &= -\widehat{\llbracket X, Y \rrbracket}, \\ \{\widehat{X}, f_M \circ \tau_*\}_{\Pi_{A^*}} &= -\rho(X)(f_M) \circ \tau_*, \\ \{f_M \circ \tau_*, h_M \circ \tau_*\}_{\Pi_{A^*}} &= 0, \end{aligned}$$

for $X, Y \in \Gamma(A)$ and $f_M, h_M \in C^\infty(M)$, $\tau_* : A^* \rightarrow M$ being the canonical projection. Here, \widehat{X} and \widehat{Y} denote the linear functions on A^* induced by the sections X and Y , respectively. Conversely, if A^* is endowed with a linear Poisson structure Π_{A^*} , then it induces a Lie algebroid structure on A characterized by (2.4) (see [4]).

Now, suppose that $(A, \llbracket \cdot, \cdot \rrbracket, \rho)$ and $(A', \llbracket \cdot, \cdot \rrbracket', \rho')$ are Lie algebroids over M and M' , respectively, and that $F : A \rightarrow A'$ is a vector bundle morphism over the map $f : M \rightarrow M'$. Then, F is said to be a *Lie algebroid morphism* if

$$(2.5) \quad d^A(F^*\alpha') = F^*(d^{A'}\alpha'), \text{ for } \alpha' \in \Gamma(\wedge^k(A')^*) \text{ and for all } k.$$

Here $F^*\alpha'$ denotes the section of the vector bundle $\wedge^k A^* \rightarrow M$ defined by

$$(2.6) \quad (F^*\alpha')_x(a_1, \dots, a_k) = \alpha'_{f(x)}(F(a_1), \dots, F(a_k)),$$

for $x \in M$ and $a_1, \dots, a_k \in A_x$.

If $F : A \rightarrow A'$ is a vector bundle isomorphism over a diffeomorphism $f : M \rightarrow M'$ then the dual isomorphism $F^* : (A')^* \rightarrow A^*$ over $f^{-1} : M' \rightarrow M$ is defined as follows

$$[F^*(\alpha'_{x'})](a_{f^{-1}(x')}) = \alpha'_{x'}(F(a_{f^{-1}(x')})),$$

for $x' \in M'$, $\alpha'_{x'} \in (A')^*_{x'}$ and $a_{f^{-1}(x')} \in A_{f^{-1}(x')}$.

Moreover, we have that F is a Lie algebroid isomorphism if and only if F^* is a Poisson isomorphism, that is,

$$\{f' \circ F^*, g' \circ F^*\}_{\Pi_{A^*}} = \{f', g'\}_{\Pi_{(A')^*}} \circ F^*, \text{ for } f', g' \in C^\infty((A')^*).$$

If F is a Lie algebroid morphism, f is an injective immersion and $F|_{A_x} : A_x \rightarrow A'_{f(x)}$ is injective, for all $x \in M$, then $(A, \llbracket \cdot, \cdot \rrbracket, \rho)$ is said to be a *Lie subalgebroid* of $(A', \llbracket \cdot, \cdot \rrbracket', \rho')$.

Let $\tilde{\pi} : A \rightarrow A'$ be an epimorphism of vector bundles over $\pi : M \rightarrow M'$, i.e. π is a submersion and for each $x \in M$, $\tilde{\pi}_x : A_x \rightarrow A'_{\pi(x)}$ is an epimorphism of vector spaces. If $X : M \rightarrow A$ is a section of A , we said that X is $\tilde{\pi}$ -projectable if there is $X' \in \Gamma(A')$ such that the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{\tilde{\pi}} & A' \\ X \uparrow & & \uparrow X' \\ M & \xrightarrow{\pi} & M' \end{array}$$

In the next proposition we will describe the necessary and sufficient conditions to obtain a Lie algebroid structure on A' such that $\tilde{\pi}$ is a Lie algebroid morphism.

Proposition 2.1. [10] *Let $(A, \llbracket \cdot, \cdot \rrbracket, \rho)$ be a Lie algebroid and $\tilde{\pi} : A \rightarrow A'$ an epimorphism of vector bundles. Then, there is a Lie algebroid structure on A' such that $\tilde{\pi}$ is a Lie algebroid epimorphism if and only if the following conditions hold:*

- (i) $\llbracket X, Y \rrbracket$ is a $\tilde{\pi}$ -projectable section of A , for all $X, Y \in \Gamma(A)$ $\tilde{\pi}$ -projectable sections of A .
- (ii) $\llbracket X, Y \rrbracket \in \Gamma(\ker \tilde{\pi})$, for all $X, Y \in \Gamma(A)$ with $X \in \Gamma(A)$ a $\tilde{\pi}$ -projectable section of A and $Y \in \Gamma(\ker \tilde{\pi})$.

An equivalent dual version of this result was proved in [3].

Let X be a section of the Lie algebroid A . The *vertical lift* of X is the vector field on A given by $X^v(a) = X(\tau(a))_a^v$ for $a \in A$, where $_a^v : A_{\tau(a)} \rightarrow T_a(A_{\tau(a)})$ is the canonical isomorphism of vector spaces.

On the other hand, there is a unique vector field X^c on A , the *complete lift* of X to A , such that X^c is τ -projectable on $\rho(X)$ and $X^c(\hat{\alpha}) = \widehat{\mathcal{L}_X^A \alpha}$, for all $\alpha \in \Gamma(A^*)$ (see [5, 6]). Here $\hat{\beta}$, with $\beta \in \Gamma(A^*)$, is the linear function on A induced by β .

We have that, for all $X, Y \in \Gamma(A)$,

$$(2.7) \quad [X^c, Y^c] = \llbracket X, Y \rrbracket^c, \quad [X^c, Y^v] = \llbracket X, Y \rrbracket^v, \quad [X^v, Y^v] = 0.$$

The flow of $X^c \in \mathfrak{X}(A)$ is related with Lie algebroid structure of A as follows.

Proposition 2.2. [7, 21] *Let $(A, \llbracket \cdot, \cdot \rrbracket, \rho)$ be a Lie algebroid over M and X a section of A . Then, for all $P \in \Gamma(\wedge^k A)$ (respectively, $\alpha \in \Gamma(\wedge^k A^*)$)*

- (i) *There exists a local flow $\varphi_s : A \rightarrow A$ which covers smooth maps $\bar{\varphi}_s : M \rightarrow M$ such that*

$$(2.8) \quad \mathcal{L}_X^A P = \frac{d}{ds}((\varphi_s)_* P)|_{s=0}, \quad (\text{respectively, } \mathcal{L}_X \alpha = \frac{d}{ds}((\varphi_s)^* \alpha)|_{s=0})$$

- (ii) $\mathcal{L}_X^A P = 0$ if and only if $(\varphi_s)_* P = P$.
- (iii) $\mathcal{L}_X^A \alpha = 0$ if and only if $\varphi_s^* \alpha = \alpha$.
- (iv) *The vector field X^c on A is complete if and only if the vector field $\rho(X)$ on M is complete.*

Here $(\varphi_s)_* P$ is the section of $\wedge^k A \rightarrow M$ defined by

$$((\varphi_s)_* P)(x)(\alpha_1, \dots, \alpha_k) = P(\bar{\varphi}_s^{-1}(x))(\varphi_s^*(\alpha_1), \dots, \varphi_s^*(\alpha_k))$$

for all $x \in M$ and $\alpha_1, \dots, \alpha_k \in A_x^*$.

If X is a section of A we define the *complete lift* of X to A^* , as the vector field X^{*c} on A^* which is τ^* -projectable on $\rho(X)$ and $X^{*c}(\hat{Y}) = \widehat{\llbracket X, Y \rrbracket}$, for all $Y \in \Gamma(A)$ (see [13]). If $\{\varphi_s\}$ is the local flow of X^c then the local flow of X^{*c} is $\{\varphi_{-s}^*\}$.

If (x^i) are local coordinates on M and $\{e_I\}$ is a local basis of sections of A , then we have the local functions ρ_I^i, C_{IJ}^K , (the *structure functions* of A) on M which are characterized by

$$\rho(e_I) = \rho_I^i \frac{\partial}{\partial x^i}, \quad \llbracket e_I, e_J \rrbracket = C_{IJ}^K e_K.$$

If (x^i, y^I) (respectively, (x^i, y_I)) denote the local coordinates on A (respectively, A^*) induced by the local basis $\{e^I\}$ (respectively, the dual basis $\{e_I\}$) then, for a section $X = X^I e_I$ of A ,

the vector fields X^v , X^c and X^{*c} are given by

$$(2.9) \quad X^v = X^I \frac{\partial}{\partial y^I}, \quad X^c = X^I \rho_I^i \frac{\partial}{\partial x^i} + (\rho_J^i \frac{\partial X^I}{\partial x^i} - X^K C_{KJ}^I) y^J \frac{\partial}{\partial y^I},$$

$$(2.10) \quad X^{*c} = X^I \rho_I^i \frac{\partial}{\partial x^i} - (\rho_I^i \frac{\partial X^K}{\partial x^i} + C_{IJ}^K X^J) y_K \frac{\partial}{\partial y^I}.$$

3. REDUCTION OF SYMPLECTIC-LIKE LIE ALGEBROIDS IN THE PRESENCE OF A MOMENTUM MAP

3.1. Actions by complete lifts for Lie algebroids. Let $(A, \llbracket \cdot, \cdot \rrbracket, \rho)$ be a Lie algebroid over the manifold M and let $\tau : A \rightarrow M$ be the corresponding vector bundle projection. We consider an left action $\Phi : G \times A \rightarrow A$ by vector bundle automorphisms of a connected Lie group G on A . Then, Φ induces a linear left action $\Phi^* : G \times A^* \rightarrow A^*$ given by

$$(\Phi_g^*)|_{A_x^*} = ((\Phi_{g^{-1}})|_{A_{\phi_g(x)}})^* : A_x^* \rightarrow A_{\phi_g(x)}^*, \quad \text{for } g \in G \text{ and } x \in M,$$

where $\phi : G \times M \rightarrow M$ is the corresponding action of G on M .

We say that $\Phi : G \times A \rightarrow A$ is an action *by complete lifts* if there is a Lie algebra anti-morphism $\psi : \mathfrak{g} \rightarrow \Gamma(A)$ such that the infinitesimal generator of $\xi \in \mathfrak{g}$ with respect to Φ is just the complete lift of $\psi(\xi)$ to A . Note that this condition implies that $\xi_M = \rho(\psi(\xi))$, where ξ_M is the infinitesimal generator of the action $\phi : G \times M \rightarrow M$ with respect to ξ . Moreover, $\psi(\xi)^c$ is a morphic vector field in the sense of [17] and therefore, for all $g \in G$, $\Phi_g : A \rightarrow A$ is a Lie algebroid automorphism. Thus, the induced action $\Phi^* : G \times A^* \rightarrow A^*$ of G on A^* is Poisson with respect to the corresponding linear Poisson structure on A^* . Furthermore, we have the following result.

Proposition 3.1. *Let $(A, \llbracket \cdot, \cdot \rrbracket, \rho)$ be a Lie algebroid on the manifold M , $\Phi : G \times A \rightarrow A$ an action by vector bundle automorphisms of a connected Lie group G on A and $\psi : \mathfrak{g} \rightarrow \Gamma(A)$ a Lie algebra anti-morphism. Then, $\Phi : G \times A \rightarrow A$ is an action of the Lie group G on A by complete lifts with respect to ψ if and only if $\Phi^* : G \times A^* \rightarrow A^*$ is an action on A^* by Poisson morphisms such that the infinitesimal generator ξ_{A^*} associated with $\xi \in \mathfrak{g}$ is just the Hamiltonian vector field corresponding to the linear function $\widehat{\psi(\xi)}$ associated with the section $\psi(\xi) \in \Gamma(A)$.*

Proof. Denote by Π_{A^*} the linear Poisson structure on A^* . We will prove that the Hamiltonian vector field

$$H_{\widehat{\psi(\xi)}}^{\Pi_{A^*}} = i_{d\widehat{\psi(\xi)}} \Pi_{A^*} \in \mathfrak{X}(A^*)$$

is just the infinitesimal generator $\xi_{A^*} \in \mathfrak{X}(A^*)$ of ξ with respect to the action Φ^* . In fact, if $f \in C^\infty(M)$ and $X \in \Gamma(A)$, using (2.4), we have that

$$\begin{aligned} H_{\widehat{\psi(\xi)}}^{\Pi_{A^*}}(f \circ \tau_*) &= \{f \circ \tau_*, \widehat{\psi(\xi)}\}_{\Pi_{A^*}} = \rho(\psi(\xi))(f) \circ \tau_* = (\psi(\xi))^*c(f \circ \tau_*) \\ H_{\widehat{\psi(\xi)}}^{\Pi_{A^*}}(\widehat{X}) &= \{\widehat{X}, \widehat{\psi(\xi)}\}_{\Pi_{A^*}} = \llbracket \widehat{\psi(\xi)}, \widehat{X} \rrbracket = (\psi(\xi))^*c(\widehat{X}). \end{aligned}$$

Here, $\{\cdot, \cdot\}_{\Pi_{A^*}}$ is the Poisson bracket associated with A^* . Thus, $H_{\widehat{\psi(\xi)}}^{\Pi_{A^*}} = (\psi(\xi))^*c$.

On the other hand, the flow of $(\psi(\xi))^c \in \mathfrak{X}(A)$ is $\{\Phi_{exp(t\xi)} : A \rightarrow A\}_{t \in \mathbb{R}}$ if and only if the flow of $(\psi(\xi))^{*c} \in \mathfrak{X}(A^*)$ is $\{\Phi_{exp(-t\xi)}^* : A^* \rightarrow A^*\}_{t \in \mathbb{R}}$ and, in consequence, we have that the proposition holds. \square

Examples 3.2. (i) If $A = TM$ and $\Phi = T\phi$ is the tangent lift of the action $\phi : G \times M \rightarrow M$ then it is clear that Φ is an action by complete lifts with Lie anti-morphism

$$\psi : \mathfrak{g} \rightarrow \mathfrak{X}(M), \quad \psi(\xi) = \xi_M.$$

(ii) Let G be Lie group. If $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ is the Lie algebra associated with G , then we have, in a natural way, a Lie algebroid structure on $\mathfrak{g} \times TM \rightarrow M$, where the Lie bracket and the anchor map are characterized by

$$[(\xi_1, X_1), (\xi_2, X_2)] = ([\xi_1, \xi_2]_{\mathfrak{g}}, [X_1, X_2]), \quad \rho(\xi, X) = X$$

for all $\xi_1, \xi_2, \xi \in \mathfrak{g}$ and $X_1, X_2, X \in \mathfrak{X}(M)$.

Now, consider a free and proper action $\phi : G \times M \rightarrow M$ of G on the manifold M . We denote by $\Phi : G \times (\mathfrak{g} \times TM) \rightarrow \mathfrak{g} \times TM$ and $\psi : \mathfrak{g} \rightarrow \Gamma(\mathfrak{g} \times TM) \cong C^\infty(M, \mathfrak{g}) \times \mathfrak{X}(M)$ the action of G on $\mathfrak{g} \times TM$ and the Lie algebra anti-morphism, respectively, given by

$$\begin{aligned} \Phi_g(\xi, v_x) &= (Ad_g^G \xi, T_x \phi_g(v_x)), \quad \xi \in \mathfrak{g} \text{ and } v_x \in T_x M \\ \psi(\bar{\xi}) &= (-\bar{\xi}, \bar{\xi}_M), \quad \bar{\xi} \in \mathfrak{g}, \end{aligned}$$

where $Ad^G : G \times \mathfrak{g} \rightarrow \mathfrak{g}$ is the adjoint action of G on \mathfrak{g} . Note that if $ad_{\bar{\xi}}^G : \mathfrak{g} \rightarrow \mathfrak{g}$ denotes the infinitesimal generator of the adjoint action for $\bar{\xi} \in \mathfrak{g}$, then the infinitesimal generator of $\bar{\xi} \in \mathfrak{g}$ with respect Φ is just $(ad_{\bar{\xi}}^G, \bar{\xi}_M^c)$. Thus, Φ is a free and proper action by complete lifts with respect to ψ .

Remark 3.3. Suppose that we have an action $\Phi : G \times A \rightarrow A$ of a Lie group G on a Lie algebroid A by complete lifts with respect to $\psi : \mathfrak{g} \rightarrow \Gamma(A)$ such that the corresponding action $\phi : G \times M \rightarrow M$ on M is free. In such a case, for all $x \in M$, $\psi_x : \mathfrak{g} \rightarrow A_x$ is injective. Indeed, if $\xi, \xi' \in \mathfrak{g}$ satisfy $\psi_x(\xi) = \psi_x(\xi')$, we have that $\xi_M(x) = \rho(\psi_x(\xi)) = \rho(\psi_x(\xi')) = \xi'_M(x)$ which implies, using the fact that ϕ is free, that $\xi = \xi'$.

Next, we will prove that each action of a connected Lie group G over a Lie algebroid A by complete lifts induces an affine action of the Lie group TG over A . Previously, we recall some facts which are related with the Lie group structure of TG .

If G is a Lie group then TG is also a Lie group. In fact, if $\cdot : G \times G \rightarrow G$ denotes the multiplication of G , then the tangent map $T\cdot : TG \times TG \rightarrow TG$ of \cdot is such that $(TG, T\cdot)$ is a Lie group. Moreover, TG may be identified with the cartesian product $G \times \mathfrak{g}$, where $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ is the corresponding Lie algebra associated with G . This identification is given by

$$TG \rightarrow G \times \mathfrak{g}, \quad X_g \in T_g G \rightarrow (g, (T_g l_{g^{-1}})(X_g)) \in G \times \mathfrak{g},$$

$l_{g^{-1}} : G \rightarrow G$ being the left translation by g^{-1} on G . The corresponding Lie group structure on $G \times \mathfrak{g}$ is defined as follows

$$(3.1) \quad (g, \xi) \cdot (g', \xi') = ((g \cdot g'), \xi' + Ad_{(g')^{-1}}^G \xi)$$

and its associated Lie algebra is $\mathfrak{g} \times \mathfrak{g}$ with the Lie bracket

$$(3.2) \quad [(\xi, \eta), (\xi', \eta')]_{\mathfrak{g} \times \mathfrak{g}} = ([\xi, \xi']_{\mathfrak{g}}, [\xi, \eta']_{\mathfrak{g}} - [\xi', \eta]_{\mathfrak{g}}).$$

Here $Ad^G : G \times \mathfrak{g} \rightarrow \mathfrak{g}$ denotes the adjoint action of G .

Moreover, if $Coad^{TG} : (G \times \mathfrak{g}) \times (\mathfrak{g}^* \times \mathfrak{g}^*) \rightarrow \mathfrak{g}^* \times \mathfrak{g}^*$ is the left coadjoint action of $TG \cong G \times \mathfrak{g}$ on the dual space of the Lie algebra $(\mathfrak{g} \times \mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g} \times \mathfrak{g}})$, then

$$(3.3) \quad Coad_{(g, \xi)}^{TG}(\mu', \mu'') = (Coad_g^G(\mu' + coad_{\xi}^G \mu''), Coad_g^G \mu''),$$

for $(g, \xi) \in G \times \mathfrak{g}$ and $(\mu', \mu'') \in \mathfrak{g}^* \times \mathfrak{g}^*$, where $Coad^G : G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is the left coadjoint action associated with G and $coad^G : \mathfrak{g} \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is the corresponding infinitesimal left coadjoint action.

The following proposition describes how ψ works with respect to the action Φ .

Proposition 3.4. *Let $\Phi : G \times A \rightarrow A$ be an action of a connected Lie group G on the Lie algebroid A by complete lifts with respect to $\psi : \mathfrak{g} \rightarrow \Gamma(A)$. Then,*

$$(3.4) \quad \Phi_g(\psi(Ad_{g^{-1}}^G \xi)(x)) = \psi(\xi)(\phi_g(x))$$

for all $\xi \in \mathfrak{g}$, $g \in G$ and $x \in M$.

Proof. We organize the proof in two steps.

First step: Suppose that G is a connected and simply connected Lie group. Consider the map

$$\psi^{cv} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{X}(A), \quad \psi^{cv}(\xi, \eta) = (\psi(\xi))^c + (\psi(\eta))^v.$$

Using (2.7), we may prove easily that ψ^{cv} is an infinitesimal action of TG over A , that is, ψ^{cv} is \mathbb{R} -linear and

$$\psi^{cv}([(\xi, \eta), (\xi', \eta')]_{\mathfrak{g} \times \mathfrak{g}}) = [\psi^{cv}(\xi, \eta), \psi^{cv}(\xi', \eta')].$$

Then, since the vector field $\psi^{cv}(\xi, \eta) \in \mathfrak{X}(A)$ is complete, from Palais Theorem (see [24]), there is a unique action $\Phi^T : TG \times A \rightarrow A$ from $TG \cong G \times \mathfrak{g}$ such that for all $(\xi, \eta) \in \mathfrak{g} \times \mathfrak{g}$

$$(\xi, \eta)_A = \psi^{cv}(\xi, \eta) = (\psi(\xi))^c + (\psi(\eta))^v.$$

Here $(\xi, \eta)_A \in \mathfrak{X}(A)$ is the infinitesimal generator of (ξ, η) with respect to the action Φ^T .

Now, suppose that $g = \exp_G(\eta)$. Then, we have that

$$(3.5) \quad \Phi_g(\psi(Ad_{g^{-1}}^G \xi)(x)) = \Phi^T((g, 0_{\mathfrak{g}}), \psi(Ad_{g^{-1}}^G \xi)(x))$$

with $0_{\mathfrak{g}}$ being the zero of \mathfrak{g} . In fact, one can prove that

$$(3.6) \quad \pi : \mathbb{R} \rightarrow G \times \mathfrak{g}, \quad s \mapsto \pi(s) = (\exp_G(s\eta), 0_{\mathfrak{g}})$$

is an one-parameter subgroup and $\frac{d\pi}{ds}|_{s=0} = (\eta, 0_{\mathfrak{g}})$. So, $\Phi^T((\exp_G(s\eta), 0_{\mathfrak{g}}), \psi(Ad_{g^{-1}}^G \xi)(x))$ is just the integral curve of $\phi^T(\eta, 0_{\mathfrak{g}}) = (\psi(\eta))^c$ at the point $\psi(Ad_{g^{-1}}^G \xi)(x) \in A_x$. In consequence,

$$\Phi^T((\exp_G(s\eta), 0_{\mathfrak{g}}), \psi(Ad_{g^{-1}}^G \xi)(x)) = \Phi(\exp_G(s\eta), \psi(Ad_{g^{-1}}^G \xi)(x)).$$

In particular, when $s = 1$, we obtain (3.5).

Furthermore,

$$(3.7) \quad \psi(Ad_{g^{-1}}^G \xi)(x) = \Phi^T((e, Ad_{g^{-1}}^G \xi), 0_x)$$

where 0_x is the zero of A_x and e is the identity element of G .

In fact, in order to prove (3.7), we consider the one-parameter subgroup

$$(3.8) \quad \pi' : \mathbb{R} \rightarrow G \times \mathfrak{g}, \quad s \mapsto \pi'(s) = (e, sAd_{g^{-1}}^G \xi).$$

Then, $\frac{d\pi'}{ds}|_{s=0} = (0_{\mathfrak{g}}, Ad_{g^{-1}}^G \xi) \in \mathfrak{g} \times \mathfrak{g}$ and, therefore, $\Phi^T((e, sAd_{g^{-1}}^G \xi), 0_x)$ is just the integral curve of $\psi^{cv}(0_{\mathfrak{g}}, Ad_{g^{-1}}^G \xi) = (\psi(Ad_{g^{-1}}^G \xi))^v \in \mathfrak{X}(A)$ at the point $0_x \in A_x$, i.e.,

$$\Phi^T((e, sAd_{g^{-1}}^G \xi), 0_x) = s\psi(Ad_{g^{-1}}^G \xi)(x).$$

In particular, if $s = 1$ we obtain (3.7).

Now, from (3.1), (3.5) and (3.7), we deduce that

$$\begin{aligned} \Phi_g(\psi(Ad_{g^{-1}}^G \xi)(x)) &= \Phi^T((g, 0_{\mathfrak{g}}), \Phi^T((e, Ad_{g^{-1}}^G \xi), 0_x)) \\ &= \Phi^T((g, 0_{\mathfrak{g}}) \cdot (e, Ad_{g^{-1}}^G \xi), 0_x) \\ &= \Phi^T((e, \xi) \cdot (g, 0_{\mathfrak{g}}), 0_x) = \Phi^T((e, \xi), \Phi^T((g, 0_{\mathfrak{g}}), 0_x)). \end{aligned}$$

On the other hand, using (3.5) (with $\xi = 0_{\mathfrak{g}}$), it follows that $\Phi^T((g, 0_{\mathfrak{g}}), 0_x) = 0_{\phi_g(x)}$. In addition, from (3.7) (with $g = e$), we obtain that $\Phi^T((e, \xi), 0_{\phi_g(x)}) = \psi(\xi)(\phi_g(x))$. This proves (3.4) for $g = \exp_G(\eta)$. Finally, using that G is connected, we conclude that (3.4) holds for all $g \in G$.

Second step: Now, we suppose that G is a connected Lie group with Lie algebra \mathfrak{g} . Denote by \tilde{G} the universal covering of G and by $\tilde{\mathfrak{g}}$ its corresponding Lie algebra. Then, the covering projection $p : \tilde{G} \rightarrow G$ is a local isomorphism of Lie groups and the map

$$\tilde{\Phi} : \tilde{G} \times A \rightarrow A, \quad \tilde{\Phi}(\tilde{g}, a_x) = \Phi(p(\tilde{g}), a_x)$$

is an action of \tilde{G} over A by complete lifts with respect to the Lie algebra anti-morphism $\tilde{\psi} = \psi \circ T_e p : \tilde{\mathfrak{g}} \rightarrow \Gamma(A)$. So, for all $g \in G$, $x \in M$ and $\xi \in \mathfrak{g}$ there are $\tilde{g} \in \tilde{G}$ and $\tilde{\xi} \in \tilde{\mathfrak{g}}$ such that

$$p(\tilde{g}) = g \text{ and } (T_{\tilde{e}} p)(\tilde{\xi}) = \xi.$$

Here, \tilde{e} is the identity element of \tilde{G} . Therefore, using the first step

$$\tilde{\Phi}_{\tilde{g}}(\tilde{\psi}(Ad_{\tilde{g}^{-1}}^{\tilde{G}} \tilde{\xi})(x)) = \psi(\xi)(\phi_g(x)).$$

Finally, since $(T_{\tilde{e}} p)(Ad_{\tilde{g}^{-1}}^{\tilde{G}} \tilde{\xi}) = Ad_{g^{-1}}^G \xi$, then we obtain (3.4). \square

As we previously claimed, from an action of G on A by complete lifts, we can define an affine action of TG on A as it is described in the following theorem.

Theorem 3.5. *Let $\Phi : G \times A \rightarrow A$ be an action of the connected Lie group G by complete lifts on the Lie algebroid A with respect to $\psi : \mathfrak{g} \rightarrow \Gamma(A)$. Then,*

$$(3.9) \quad \Phi^T : (G \times \mathfrak{g}) \times A \rightarrow A, \quad \Phi^T((g, \xi), a_x) = \Phi_g(a_x + \psi(\xi)(x))$$

defines an affine action of $TG \cong G \times \mathfrak{g}$ over A . Moreover, if $(\xi, \eta) \in \mathfrak{g} \times \mathfrak{g}$, its infinitesimal generator $(\xi, \eta)_A$ with respect to the action Φ^T is

$$(3.10) \quad (\xi, \eta)_A = (\psi(\xi))^c + (\psi(\eta))^v.$$

Proof. Equation (3.4) allows to prove that $\Phi^T : (G \times \mathfrak{g}) \times A \rightarrow A$ is an affine action of $TG \cong G \times \mathfrak{g}$ over A . In fact,

$$\begin{aligned} \Phi^T((g, \xi) \cdot (h, \eta), a_x) &= \Phi^T((g \cdot h, \eta + \text{Ad}_{h^{-1}}^G \xi), a_x) = \Phi_{gh}(a_x) + \Phi_{gh}(\psi(\eta + \text{Ad}_{h^{-1}}^G \xi)(x)) \\ &= \Phi_g(\Phi_h(a_x) + \Phi_h(\psi(\eta)(x) + \psi(\text{Ad}_{h^{-1}}^G \xi)(x))) \\ &= \Phi_g(\Phi_h(a_x) + \Phi_h(\psi(\eta)(x))) + \Phi_g(\psi(\xi)(\phi_h(x))) \\ &= \Phi^T((g, \xi), \Phi^T((h, \eta), a_x)) \end{aligned}$$

and

$$\Phi^T((e, 0_{\mathfrak{g}}), a_x) = \Phi_e(a_x) + \Phi_e(\psi(0_{\mathfrak{g}})(x)) = a_x.$$

Moreover, using the one-parameter subgroups defined in (3.6) and (3.8), one may conclude easily that the infinitesimal generator $(\xi, \eta)_A$ of $(\xi, \eta) \in \mathfrak{g} \times \mathfrak{g}$ with respect Φ^T is

$$(\xi, \eta)_A = (\xi, 0_{\mathfrak{g}})_A + (0_{\mathfrak{g}}, \eta)_A = (\psi(\xi))^c + (\psi(\eta))^v.$$

□

Note that, under the same hypotheses as in Theorem 3.5, the action $\Phi : G \times A \rightarrow A$ is free and proper if and only if the corresponding action $\phi : G \times M \rightarrow M$ on M is free and proper. Moreover, if $\Phi : G \times A \rightarrow A$ is free and proper then so is $\Phi^T : TG \times A \rightarrow A$.

On the other hand, we recall that the space of orbits N/H of a free and proper action of a Lie group H on a manifold N is a quotient differentiable manifold and the canonical projection $\pi : N \rightarrow N/H$ is a surjective submersion (see [1]). With this, we prove a preliminary reduction result.

Theorem 3.6. *Let $(A, [\cdot, \cdot], \rho)$ be a Lie algebroid on M and $\Phi : G \times A \rightarrow A$ a free and proper action of a connected Lie group G on A by complete lifts with respect to the Lie algebra anti-morphism $\psi : \mathfrak{g} \rightarrow \Gamma(A)$. Then, A/TG is a Lie algebroid over M/G and the projection $\tilde{\pi} : A \rightarrow A/TG$ is a Lie algebroid epimorphism.*

Proof. Since that $\Phi : G \times A \rightarrow A$ is an action by Lie algebroid automorphisms then A/G is a Lie algebroid over M/G with vector bundle projection $\tau/G : A/G \rightarrow M/G$ (see [10, 15]). The space of sections of this vector bundle may be identified with the one of G -invariant sections $\Gamma(A)^G$ of A . Under this identification the bracket and the anchor map of the Lie algebroid structure on A/G is just

$$[[X, Y]]_{A/G} = [[X, Y]], \quad \rho_{A/G}(X(\pi(x))) = T_x \pi(\rho(X(x)))$$

for all $X, Y \in \Gamma(A)^G$ and $x \in M$, where $\pi : M \rightarrow M/G$ is the quotient projection corresponding to the induced action $\phi : G \times M \rightarrow M$.

On the other hand, using that $\phi : G \times M \rightarrow M$ is free, then we have that $\psi_x : \mathfrak{g} \rightarrow A_x$, is injective (see Remark 3.3). Thus, for all $x \in M$, we have that

$$\dim\{\psi_x(\xi)/\xi \in \mathfrak{g}\} = \dim \mathfrak{g}, \text{ for all } x \in M.$$

Therefore, since $\psi : \mathfrak{g} \rightarrow \Gamma(A)$ is a Lie algebra anti-morphism, we deduce that

$$\psi(\mathfrak{g}) = \bigcup_{x \in M} \{\psi_x(\xi)/\xi \in \mathfrak{g}\}$$

is a Lie subalgebroid of A over M . Moreover, from Proposition 3.4, we have that the Lie group G acts by Lie algebroid automorphisms on $\psi(\mathfrak{g})$. So, one may induce a Lie algebroid structure on the quotient vector bundle $\psi(\mathfrak{g})/G$ such that $\psi(\mathfrak{g})/G$ is a Lie subalgebroid of A/G . Now, we will show that it is also an ideal.

If $X \in \Gamma(A)$ is G -invariant then $X \circ \phi_g = \Phi_g \circ X$, for all $g \in G$. Thus, the flow $\Upsilon_t(\xi) : A \rightarrow A$ of the vector field $(\psi(\xi))^c$ and the flow $\varphi_t(\xi) : M \rightarrow M$ of $(\rho \circ \psi)(\xi)$ satisfy the following property

$$\Upsilon_t(\xi) \circ X = X \circ \varphi_t(\xi), \text{ for all } t \in \mathbb{R}.$$

Equivalently,

$$\widehat{X} \circ \Upsilon_t(\xi)^* = \widehat{X}$$

where $\Upsilon_t(\xi)^*$ is the dual morphism of $\Upsilon_t(\xi) : A \rightarrow A$ and \widehat{X} is the linear function associated with the section X .

Therefore, we have that

$$\frac{d}{dt}\bigg|_{t=0} (\widehat{X} \circ \Upsilon_t(\xi)^*) = 0.$$

Since $\Upsilon_t(\xi)^*$ is the flow of the vector field $\psi(\xi)^{*c}$, the previous equation is equivalent to the relation

$$(3.11) \quad \llbracket \psi(\xi), X \rrbracket = 0, \text{ for all } \xi \in \mathfrak{g}.$$

On the other hand, let Y be a G -invariant section of $\psi(\mathfrak{g})$ and $\{\xi_i\}$ a basis of \mathfrak{g} . Then,

$$Y = \sum Y^i \psi(\xi_i),$$

with Y^i real functions on A . Moreover, using Proposition 3.4 and the fact that ψ_x is injective, we have that

$$(3.12) \quad Y^i \circ \phi_g = (Ad^G)_j^i(g) Y^j,$$

where $Ad^G \xi_i = (Ad^G)_i^j(g) \xi_j$. Hence, from (3.11),

$$\llbracket X, Y \rrbracket = \sum \rho(X)(Y^i) \psi(\xi_i).$$

Now, using that X and Y are G -invariant sections, we obtain that $\llbracket X, Y \rrbracket$ is a G -invariant section and, as a consequence, $\llbracket X, Y \rrbracket$ is a G -invariant section of the vector bundle $\psi(\mathfrak{g}) \rightarrow M$. Thus, $\psi(\mathfrak{g})/G$ is indeed an ideal of A/G of constant rank (since the action Φ is free) and therefore, the quotient vector bundle $(A/G)/(\psi(\mathfrak{g})/G)$ admits a Lie algebroid structure over M/G .

Finally, we have that this vector bundle is isomorphic to A/TG and, thus, a Lie algebroid structure on A/TG is induced in such a way that this isomorphism is a Lie algebroid isomorphism. In fact, using (3.4), one may prove that Φ induces a free and proper action $\bar{\Phi} : G \times A/\psi(\mathfrak{g}) \rightarrow A/\psi(\mathfrak{g})$ on $A/\psi(\mathfrak{g})$ such that the projection $\tilde{\Psi}_1 : A \rightarrow A/\psi(\mathfrak{g})$ is equivariant, with respect to the action Φ^T and $\bar{\Phi}$. So, $\tilde{\Psi}_1$ induces a smooth map $\Psi_1 : A/TG \rightarrow (A/\psi(\mathfrak{g}))/G$. Moreover, one easily proves that Ψ_1 is a one-to-one correspondence. On the other hand, the map

$$\Psi_2 : (A/\psi(\mathfrak{g}))/G \rightarrow (A/G)/(\psi(\mathfrak{g})/G), \quad [\tilde{\Psi}_1(a)] \rightarrow \tilde{\Psi}_2([a])$$

is bijective, where $\tilde{\Psi}_2 : A/G \rightarrow (A/G)/(\psi(\mathfrak{g})/G)$ is the corresponding quotient map. Consequently the reduced vector bundle $A/TG \rightarrow M/G$ is isomorphic to the vector bundles

$$(3.13) \quad (A/\psi(\mathfrak{g}))/G \rightarrow M/G \quad \text{and} \quad (A/G)/(\psi(\mathfrak{g})/G) \rightarrow M/G$$

Note that, using the above isomorphisms, the space of sections of the vector bundle $A/TG \rightarrow M/G$ may be identified with the quotient space

$$\Gamma(A)^G / \Gamma(\psi(\mathfrak{g}))^G$$

where $\Gamma(A)^G$ (respectively, $\Gamma(\psi(\mathfrak{g}))^G$) is the space of G -invariant sections on A (respectively, $\psi(\mathfrak{g})$). Under this identification the Lie algebroid structure $([\![\cdot, \cdot]\!]_{A/TG}, \rho_{A/TG})$ on A/TG is characterized by

$$(3.14) \quad \begin{aligned} [[X], [Y]]_{A/TG} &= [[X, Y]], \\ \rho_{A/TG}([X]) \circ \pi &= T\pi \circ \rho(X), \quad \text{for } X, Y \in \Gamma(A)^G. \end{aligned}$$

This implies that the canonical projection $\tilde{\pi} : A \rightarrow A/TG$ is a Lie algebroid epimorphism. \square

Examples 3.7. (i) In the case when $A = TM$ and $\Phi = T\phi$ is the tangent lift of the action $\phi : G \times M \rightarrow M$, the reduced Lie algebroid TM/TG from the previous theorem is isomorphic to $T(M/G)$ with its standard Lie algebroid structure.

(ii) For the case $A = \mathfrak{g} \times TM$ from (ii) in Examples 3.2, we obtain that the reduced Lie algebroid $(\mathfrak{g} \times TM)/TG$ with respect to the action $\Phi^T : (G \times \mathfrak{g}) \times (\mathfrak{g} \times TM) \rightarrow \mathfrak{g} \times TM$ given by

$$\Phi^T((g, \bar{\xi}), (\xi, v_x)) = (Ad_g^G(\xi - \bar{\xi}), T_x\phi_g(v_x + \bar{\xi}_M(x)))$$

can be identified with the Atiyah Lie algebroid TM/G induced by the principal bundle $\pi : M \rightarrow M/G$.

We recall the construction of this last Lie algebroid. Firstly, we denote by $\tau : TM \rightarrow M$ the projection of TM on M which is equivariant with respect to the tangent lift action $T\phi : G \times TM \rightarrow TM$ and $\phi : G \times M \rightarrow M$. The sections of the induced vector bundle $\tau/G : TM/G \rightarrow M/G$ can be identified with the G -invariant vector fields on M . Moreover, the set of G -invariant vector fields is closed with respect to the Lie bracket of vector fields. Using this fact, one can define the Lie algebroid structure $([\![\cdot, \cdot]\!]_{TM/G}, \rho_{TM/G})$ on TM/G by

$$[\![X, Y]\!]_{TM/G} = [X, Y], \quad \rho_{TM/G}(X(x)) = T_x\pi(X(x)),$$

for X, Y G -invariant vector fields of M and $x \in M$. The corresponding Lie algebroid is known as *Atiyah Lie algebroid* (see [13]).

Now, we have the following vector bundle epimorphism

$$(3.15) \quad F : (\mathfrak{g} \times TM)/TG \rightarrow TM/G, \quad F([\!(\xi, v_x)\!]_{TG}) = [v_x + \xi_M]_G.$$

Note that F is well-defined. Indeed, for all $\xi, \bar{\xi} \in \mathfrak{g}$, $g \in G$ and $v_x \in T_xM$ one has that

$$\begin{aligned} F([Ad_g^G(\xi - \bar{\xi}), T_x\phi_g(v_x + \bar{\xi}_M(x))]_{TG}) &= [T_x\phi_g(v_x + \bar{\xi}_M(x)) + (Ad_g^G(\xi - \bar{\xi}))_M(\phi_g(x))]_G \\ &= [T_x\phi_g(v_x + \bar{\xi}_M(x)) + T_x\phi_g(\xi_M(x) - \bar{\xi}_M(x))]_G \\ &= [v_x + \xi_M(x)]_G. \end{aligned}$$

Moreover, since

$$(3.16) \quad [(\xi, v_x)]_{TG} = [(0, v_x + \xi_M(x))]_{TG}, \quad \text{for all } v_x \in T_x M \text{ and } \xi \in \mathfrak{g},$$

we deduce that F is a vector bundle isomorphism.

On the other hand, using (3.16), we obtain that if $\{X_i\}$ is a local basis of G -invariant vector fields on M then $\{[(0, X_i)]_{TG}\}$ is a local base of $\Gamma((\mathfrak{g} \times TM)/TG)$. This fact allows to prove that F is a Lie algebroid isomorphism.

3.2. Reduction of Symplectic Lie algebroids. A Lie algebroid $(A, [\cdot, \cdot], \rho)$ on the manifold M is *symplectic-like* if there is a nondegenerate 2-section $\Omega \in \Gamma(\wedge^2 A^*)$ on A^* which is closed, i.e. $d^A \Omega = 0$. In such a case, for each function $f : M \rightarrow \mathbb{R}$ on M , we have the *Hamiltonian section* on A which is characterized by

$$i_{\mathcal{H}_f^\Omega} \Omega = d^A f.$$

The base space M of a symplectic-like Lie algebroid A is a Poisson manifold, where the Poisson bracket on M is given by

$$(3.17) \quad \{f, g\} = \rho(\mathcal{H}_g^\Omega)(f) \quad f, g \in C^\infty(M).$$

(see [13, 11, 16]).

Note that if $f \in C^\infty(M)$ then the Hamiltonian vector field of f with respect to the Poisson structure on M is $\rho(\mathcal{H}_f^\Omega)$. Thus, the solutions of Hamilton's equations for f are the integral curves of the vector field $\rho(\mathcal{H}_f^\Omega)$.

Now, in the rest of this section, we suppose that $(A, [\cdot, \cdot], \rho, \Omega)$ is a symplectic-like Lie algebroid over M , that $\Phi : G \times A \rightarrow A$ is an action of a connected Lie group G on A by complete lifts with respect to the Lie algebra anti-morphism $\psi : \mathfrak{g} \rightarrow \Gamma(A)$ and that $J : M \rightarrow \mathfrak{g}^*$ is an equivariant smooth map, i.e.,

$$\text{Coad}_g^G(J(x)) = J(\phi_g(x)), \quad \forall x \in M, \quad \forall g \in G.$$

The action Φ is said to be a *Hamiltonian action with momentum map* $J : M \rightarrow \mathfrak{g}^*$ if

$$(3.18) \quad \Phi_g^*(\Omega) = \Omega \text{ and } i_{\psi(\xi)} \Omega = d^A J_\xi, \text{ for all } g \in G \text{ and } \xi \in \mathfrak{g},$$

where J_ξ is the real function on M given by

$$(3.19) \quad J_\xi(x) = J(x)(\xi), \text{ for } x \in M.$$

Note that the previous condition implies that

$$(3.20) \quad \mathcal{L}_{\psi(\xi)}^A \Omega = 0.$$

Now, let $J^T : A \rightarrow (\mathfrak{g} \times \mathfrak{g})^* \cong \mathfrak{g}^* \times \mathfrak{g}^*$ be the map given by

$$(3.21) \quad J^T(a) = ((TJ \circ \rho)(a), J(\tau(a))).$$

Then we have

Lemma 3.8. *The map $J^T : A \rightarrow \mathfrak{g}^* \times \mathfrak{g}^*$ is equivariant for the action $\Phi^T : TG \times A \rightarrow A$.*

Proof. Let $(g, \xi) \in G \times \mathfrak{g} \cong TG$ and $a \in A_x$. Since J is equivariant, we have that

$$(3.22) \quad T_x J(\rho(\psi(\xi)(x))) = \text{coad}_\xi^G(J(x)), \quad \text{for } \xi \in \mathfrak{g} \text{ and } x \in M.$$

Moreover, using that Φ_g is a Lie algebroid morphism over ϕ_g and that J is equivariant, we obtain

$$(3.23) \quad TJ \circ \rho \circ \Phi_g = TJ \circ T\phi_g \circ \rho = \text{Coad}_g^G \circ TJ \circ \rho.$$

As a consequence, from (3.3), (3.21), (3.22) and (3.23), we have that

$$\begin{aligned} \text{Coad}_{(g,\xi)}^{TG}(J^T(a)) &= \text{Coad}_{(g,\xi)}^{TG}(TJ(\rho(a)), J(\tau(a))) \\ &= (\text{Coad}_g^G(TJ \circ \rho(a)) + \text{Coad}_g^G(\text{coad}_\xi^G J(\tau(a))), \text{Coad}_g^G(J(\tau(a)))) \\ &= (\text{Coad}_g^G(TJ \circ \rho(a)) + \text{Coad}_g^G(\text{coad}_\xi^G J(\tau(a))), J(\phi_g(\tau(a)))) \\ &= ((TJ \circ \rho)(\Phi_g(a))) + (TJ \circ \rho)(\Phi_g(\psi(\xi)(\tau(a)))), J(\tau(\Phi_{(g,\xi)}^T(a)))) \\ &= ((TJ \circ \rho)(\Phi_{(g,\xi)}^T(a)), J(\tau(\Phi_{(g,\xi)}^T(a)))) \\ &= J^T(\Phi_{(g,\xi)}^T(a)). \end{aligned}$$

Hence J^T is equivariant with respect to Φ^T . \square

Proposition 3.9. *Let $(A, \llbracket \cdot, \cdot \rrbracket, \rho, \Omega)$ be a symplectic-like Lie algebroid over M , $\Phi : G \times A \rightarrow A$ a Hamiltonian action of a connected Lie group G on A with Lie algebra anti-morphism $\psi : \mathfrak{g} \rightarrow \Gamma(A)$ and equivariant momentum map $J : M \rightarrow \mathfrak{g}^*$. Let $\mu \in \mathfrak{g}^*$ be a regular value of J such that $T_x J \circ \rho : A_x \rightarrow T_\mu \mathfrak{g}^*$ has constant rank for all $x \in J^{-1}(\mu)$. Then,*

- (i) $(J^T)^{-1}(0, \mu)$ is a Lie subalgebroid of A over $J^{-1}(\mu)$.
- (ii) The restriction $\psi_\mu : \mathfrak{g}_\mu \rightarrow \Gamma(A)$ of ψ to the isotropy algebra \mathfrak{g}_μ of μ with respect to the coadjoint action takes values in $\Gamma((J^T)^{-1}(0, \mu))$.
- (iii) The isotropy Lie group G_μ of μ with respect to the coadjoint action acts on $(J^T)^{-1}(0, \mu)$ by complete lifts with respect to $\psi_\mu : \mathfrak{g}_\mu \rightarrow \Gamma((J^T)^{-1}(0, \mu))$.
- (iv) The action of G_μ on the Lie subalgebroid $(J^T)^{-1}(0, \mu)$ induces an affine action $\Phi_\mu^T : TG_\mu \times (J^T)^{-1}(0, 0) \rightarrow (J^T)^{-1}(0, 0)$.

Proof. (i) Note that since μ is a regular value of J , $J^{-1}(\mu)$ is a regular submanifold of M . In fact, $(0, \mu)$ is a regular value for $J^T : A \rightarrow \mathfrak{g}^* \times \mathfrak{g}^* \cong T\mathfrak{g}^*$. Thus, using that $T_x J \circ \rho_x : A_x \rightarrow T_\mu \mathfrak{g}^*$ has constant rank for all $x \in J^{-1}(\mu)$, we deduce that

$$(J^T)^{-1}(0, \mu) = \{a \in A / TJ(\rho(a)) = 0, \quad J(\tau(a)) = \mu\}$$

is a vector subbundle of A on $J^{-1}(\mu)$ of rank $n - \dim G$, where $n = \text{rank } A$. On the other hand, from (3.21) we have that J^T is a Lie algebroid morphism then it is straightforward to see that the restriction $\tau_{(J^T)^{-1}(0, \mu)} : (J^T)^{-1}(0, \mu) \rightarrow J^{-1}(\mu)$ of $\tau : A \rightarrow M$ to $(J^T)^{-1}(0, \mu)$ is a Lie subalgebroid of $(A, \llbracket \cdot, \cdot \rrbracket, \rho)$.

(ii) If $x \in J^{-1}(\mu)$ and

$$\xi \in \mathfrak{g}_\mu = \{\eta \in \mathfrak{g} / \text{coad}_\eta^G \mu = 0\}$$

then, since J is an equivariant map, we have that

$$T_x J(\rho_x(\psi(\xi)(x))) = T_x J(\xi_M(x)) = \text{coad}_\xi^G(\mu) = 0.$$

Thus, the restriction of $\psi(\xi)$ to $J^{-1}(\mu)$ is a section of the vector bundle $(J^T)^{-1}(0, \mu) \rightarrow J^{-1}(\mu)$.

(iii) Using the equivariance of J and the fact that Φ_g is a Lie algebroid automorphism, for any $g \in G$ we have that the action $\Phi : G \times A \rightarrow A$ induces an action Φ_μ of G_μ on $(J^T)^{-1}(0, \mu)$. In fact, if $g \in G_\mu$, $a \in (J^T)^{-1}(0, \mu)$ and $x \in J^{-1}(\mu)$, then

$$J(\phi_g(x)) = \text{Coad}_g^G(J(x)) = \text{Coad}_g^G \mu = \mu$$

and

$$(TJ \circ \rho)(\Phi_g(a)) = T(J \circ \phi_g)(\rho(a)) = T(\text{Coad}_g^G \circ J)(\rho(a)) = 0.$$

Moreover, from (ii), we have that Φ_μ is an action by complete lifts with respect to the Lie algebra anti-morphism ψ_μ .

(iv) It is a direct consequence of (iii) and Theorem 3.5. \square

Let $\mu \in \mathfrak{g}^*$ be a regular value of $J : M \rightarrow \mathfrak{g}^*$ such that $T_x J \circ \rho_x : A_x \rightarrow T_\mu \mathfrak{g}^*$ has constant rank for all $x \in J^{-1}(\mu)$. Suppose that the corresponding action $\phi_\mu : G_\mu \times J^{-1}(\mu) \rightarrow J^{-1}(\mu)$ is free and proper. Then, using Theorem 3.6 and Proposition 3.9, we obtain that $A_\mu = (J^T)^{-1}(0, \mu)/TG_\mu$ is a Lie algebroid over $J^{-1}(\mu)/G_\mu$. In the following result, we will prove that A_μ is a symplectic-like Lie algebroid. For this purpose, we will need the following properties.

Lemma 3.10. *Let $(A, [\cdot, \cdot], \rho, \Omega)$ be a symplectic-like Lie algebroid over the manifold M and $\Phi : G \times A \rightarrow A$ a Hamiltonian action of a connected Lie group G on A with equivariant momentum map $J : M \rightarrow \mathfrak{g}^*$ and associated Lie algebra anti-morphism $\psi : \mathfrak{g} \rightarrow \Gamma(A)$. If $\mu \in \mathfrak{g}^*$, then for any $x \in M$,*

- (i) $(\psi_\mu)_x(\mathfrak{g}_\mu) = \psi_x(\mathfrak{g}) \cap \ker(T_x J \circ \rho_x)$
- (ii) $\ker(T_x J \circ \rho_x) = (\psi_x(\mathfrak{g}))^\perp = \{a_x \in A_x / \Omega_x(a_x, b_x) = 0, \forall b_x \in \psi_x(\mathfrak{g})\}.$

Proof. (i) It is an immediate consequence of the fact that J is equivariant.

(ii) If $a_x \in A_x$, using (2.3) and (3.19), we deduce that

$$\Omega(a_x, \psi(\xi)(x)) = -(i_{\psi(\xi)} \Omega)(a_x) = -(d^A J_\xi)(a_x) = -(T_x J(\rho_x(a_x)))(\xi),$$

for all $\xi \in \mathfrak{g}$. Thus, one concludes immediately (ii) from this relation. \square

The following result may be seen as the analogous of Marsden-Weinstein reduction Theorem for symplectic-like Lie algebroids.

Theorem 3.11. Reduction Theorem of symplectic-like Lie algebroids *Let $(A, [\cdot, \cdot], \rho, \Omega)$ be a symplectic-like Lie algebroid and $\Phi : G \times A \rightarrow A$ a Hamiltonian action of a connected Lie group G on A with equivariant momentum map $J : M \rightarrow \mathfrak{g}^*$ and associated Lie algebra anti-morphism $\psi : \mathfrak{g} \rightarrow \Gamma(A)$. Suppose that $\mu \in \mathfrak{g}^*$ is a regular value of $J : M \rightarrow \mathfrak{g}^*$ such that $T_x J \circ \rho_x : A_x \rightarrow T_\mu \mathfrak{g}^*$ has constant rank for all $x \in J^{-1}(\mu)$ and the restricted action $\phi_\mu : G_\mu \times J^{-1}(\mu) \rightarrow J^{-1}(\mu)$ is free and proper. Then $A_\mu = (J^T)^{-1}(0, \mu)/TG_\mu$ is a symplectic-like Lie algebroid over $J^{-1}(\mu)/G_\mu$ with symplectic-like section Ω_μ characterized by the condition*

$$\tilde{\pi}_\mu^* \Omega_\mu = \tilde{\iota}_\mu^* \Omega,$$

where $\tilde{\pi}_\mu : (J^T)^{-1}(0, \mu) \rightarrow A_\mu$ is the canonical projection and $\tilde{\iota}_\mu : (J^T)^{-1}(0, \mu) \rightarrow A$ is the canonical inclusion.

Proof. Since A_μ is a Lie algebroid over $J^{-1}(\mu)/G_\mu$ then one needs to prove that this algebroid is symplectic-like.

Let $\tilde{\Omega}_\mu = \tilde{\iota}_\mu^* \Omega$ be the 2-cocycle on the Lie subalgebroid $(J^T)^{-1}(0, \mu) \rightarrow J^{-1}(\mu)$ induced by Ω .

We will prove that $\tilde{\Omega}_\mu$ induces a symplectic-like 2-section Ω_μ over A_μ .

Suppose that $X_\mu, Y_\mu \in \Gamma(A_\mu)$. Then, we may choose two sections $\tilde{X}_\mu, \tilde{Y}_\mu \in \Gamma((J^T)^{-1}(0, \mu))$ such that the following diagram is commutative

$$\begin{array}{ccc} J^{-1}(\mu) & \xrightarrow{\tilde{X}_\mu, \tilde{Y}_\mu} & (J^T)^{-1}(0, \mu) \\ \pi_\mu \downarrow & & \downarrow \tilde{\pi}_\mu \\ J^{-1}(\mu)/G_\mu & \xrightarrow{X_\mu, Y_\mu} & A_\mu \end{array}$$

We will see that $\tilde{\Omega}_\mu(\tilde{X}_\mu, \tilde{Y}_\mu)$ is a G_μ -invariant function (or, equivalently, a π_μ -basic function).

Denote by $(\llbracket \cdot, \cdot \rrbracket_{(J^T)^{-1}(0, \mu)}, \rho_{(J^T)^{-1}(0, \mu)})$ the Lie algebroid structure on $(J^T)^{-1}(0, \mu) \rightarrow J^{-1}(\mu)$.

As we know, the vertical bundle of π_μ is generated by the vector fields on $J^{-1}(\mu)$ of the form

$\rho_{(J^T)^{-1}(0, \mu)}(\psi_\mu(\xi))$, with $\xi \in \mathfrak{g}_\mu$.

Now, we have that

$$\begin{aligned} (\rho_{(J^T)^{-1}(0, \mu)}(\psi_\mu(\xi)))(\tilde{\Omega}_\mu(\tilde{X}_\mu, \tilde{Y}_\mu)) &= (\mathcal{L}_{\psi_\mu(\xi)}^{(J^T)^{-1}(0, \mu)} \tilde{\Omega}_\mu)(\tilde{X}_\mu, \tilde{Y}_\mu) + \tilde{\Omega}_\mu(\llbracket \psi_\mu(\xi), \tilde{X}_\mu \rrbracket_{(J^T)^{-1}(0, \mu)}, \tilde{Y}_\mu) \\ &\quad + \tilde{\Omega}_\mu(\tilde{X}_\mu, \llbracket \psi_\mu(\xi), \tilde{Y}_\mu \rrbracket_{(J^T)^{-1}(0, \mu)}). \end{aligned}$$

On the other hand, using that \tilde{X}_μ and \tilde{Y}_μ are G_μ -invariant, we deduce that

$$\llbracket \psi_\mu(\xi), \tilde{X}_\mu \rrbracket_{(J^T)^{-1}(0, \mu)} = \llbracket \psi_\mu(\xi), \tilde{Y}_\mu \rrbracket_{(J^T)^{-1}(0, \mu)} = 0$$

(see (3.11)). In addition, from (3.20), it follows that

$$\mathcal{L}_{\psi_\mu(\xi)}^{(J^T)^{-1}(0, \mu)} \tilde{\Omega}_\mu = 0,$$

which implies that

$$\rho_{(J^T)^{-1}(0, \mu)}(\psi_\mu(\xi))(\tilde{\Omega}_\mu(\tilde{X}_\mu, \tilde{Y}_\mu)) = 0.$$

Thus, for all $X_\mu, Y_\mu \in \Gamma(A_\mu)$ there is a function $\Omega_\mu(X_\mu, Y_\mu)$ on $J^{-1}(\mu)/G_\mu$ such that

$$\Omega_\mu(X_\mu, Y_\mu) \circ \pi_\mu = \tilde{\Omega}_\mu(\tilde{X}_\mu, \tilde{Y}_\mu).$$

Note that the function $\Omega_\mu(X_\mu, Y_\mu)$ does not depend on the choosen sections $\tilde{X}_\mu, \tilde{Y}_\mu \in \Gamma((J^T)^{-1}(0, \mu))$ which project on \tilde{X}_μ and \tilde{Y}_μ , respectively. In fact, from Lemma 3.10, we have that

$$\ker(\tilde{\Omega}_\mu(x)) = (\ker \tilde{\pi}_\mu)|_{(J^T)^{-1}_x(0, \mu)} = (\psi_\mu)_x(\mathfrak{g}_\mu) \text{ for all } x \in J^{-1}(\mu).$$

Therefore, the map

$$\Gamma(A_\mu) \times \Gamma(A_\mu) \rightarrow C^\infty(J^{-1}(\mu)/G_\mu), \quad (X_\mu, Y_\mu) \mapsto \Omega_\mu(X_\mu, Y_\mu)$$

defines a section Ω_μ of the vector bundle $\Lambda^2 A_\mu^* \rightarrow J^{-1}(\mu)/G_\mu$ and it is clear that

$$\tilde{\pi}_\mu^* \Omega_\mu = \tilde{\iota}_\mu^* \Omega = \tilde{\Omega}_\mu.$$

This implies that

$$\tilde{\pi}_\mu^*(d^{A_\mu}\Omega_\mu) = d^{(J^T)^{-1}(0,\mu)}(\tilde{\pi}_\mu^*\Omega_\mu) = d^{(J^T)^{-1}(0,\mu)}\tilde{\iota}_\mu^*\Omega = \tilde{\iota}_\mu^*(d^A\Omega) = 0$$

and, since $\tilde{\pi}_\mu : (J^T)^{-1}(0,\mu) \rightarrow A_\mu = (J^T)^{-1}(0,\mu)/TG_\mu$ is an epimorphism of vector bundles, we conclude that $d^{A_\mu}\Omega_\mu = 0$.

Finally, we will prove that Ω_μ is non-degenerate. In fact, if $x \in J^{-1}(\mu)$ and $\tilde{v}_x \in \ker(T_x J \circ \rho_x)$ is such that

$$\Omega_\mu(\pi_\mu(x))(\tilde{\pi}_\mu(\tilde{v}_x), u_{\pi_\mu(x)}) = 0, \quad \forall u_{\pi_\mu(x)} \in (A_\mu)_{\pi_\mu(x)}$$

then

$$\tilde{\Omega}(x)(\tilde{v}_x, \tilde{u}_x) = 0 \quad \text{for all } \tilde{u}_x \in \ker(T_x J \circ \rho_x).$$

Consequently (see Lemma 3.10)

$$\tilde{v}_x \in (\ker(T_x J \circ \rho_x))^\perp = \psi_x(\mathfrak{g})$$

and thus,

$$\tilde{v}_x \in (\psi_\mu)_x(\mathfrak{g}_\mu)$$

which implies that $\tilde{\pi}_\mu(\tilde{v}_x) = 0$.

□

Remark 3.12. In the particular case when M is a symplectic manifold and A is the standard symplectic-like Lie algebroid $TM \rightarrow M$ then Theorem 3.11 reproduces the classical Marsden-Weinstein reduction result for the symplectic manifold M .

Since $A_\mu \rightarrow J^{-1}(\mu)/G_\mu$ is a symplectic-like Lie algebroid, the base space $J^{-1}(\mu)/G_\mu$ is a Poisson manifold. In fact, we will prove that $J^{-1}(\mu)/G_\mu$ is the reduced Poisson manifold $(M, \{\cdot, \cdot\})$ obtained from the reduction process of Marsden-Ratiu [19].

Theorem 3.13. *Under the hypotheses of Theorem 3.11, if $\{\cdot, \cdot\}_\mu$ is the Poisson bracket on $J^{-1}(\mu)/G_\mu$, we have that*

$$(3.24) \quad \{\tilde{f}, \tilde{g}\}_\mu \circ \pi_\mu = \{f, g\} \circ i_\mu$$

for $\tilde{f}, \tilde{g} \in C^\infty(J^{-1}(\mu)/G_\mu)$, where $i_\mu : J^{-1}(\mu) \rightarrow M$ is the canonical inclusion and $f, g \in C^\infty(M)$ are arbitrary G -invariant extensions of $\tilde{f} \circ \pi_\mu$ and $\tilde{g} \circ \pi_\mu$, respectively.

Proof. From Theorem 3.11, we deduce that $(A_\mu, [\cdot, \cdot]_{A_\mu}, \rho_{A_\mu}, \Omega_\mu)$ is a symplectic-like Lie algebroid on $J^{-1}(\mu)/G_\mu$. Then one can define a Poisson structure on $J^{-1}(\mu)/G_\mu$ as in (3.17). We will prove that the associated Poisson bracket $\{\cdot, \cdot\}_\mu$ satisfies (3.24).

If $\tilde{f}, \tilde{g} : J^{-1}(\mu)/G_\mu \rightarrow \mathbb{R}$ are two real functions on $J^{-1}(\mu)/G_\mu$ and $f : M \rightarrow \mathbb{R}$, $g : M \rightarrow \mathbb{R}$ are arbitrary G -invariant extensions of $\tilde{f} \circ \pi_\mu$ and $\tilde{g} \circ \pi_\mu$, respectively, for any $\xi \in \mathfrak{g}$ satisfying $\rho(\psi(\xi))(f) = \rho(\psi(\xi))(g) = 0$, we have that

$$d^A f(\psi(\xi)) = d^A g(\psi(\xi)) = 0,$$

or, equivalently,

$$\Omega(\mathcal{H}_f^\Omega, \psi(\xi)) = \Omega(\mathcal{H}_g^\Omega, \psi(\xi)) = 0.$$

Therefore, $\mathcal{H}_f^\Omega(x), \mathcal{H}_g^\Omega(x) \in \psi_x(\mathfrak{g})^\perp = (J^T)_x^{-1}(0, \mu)$, for all $x \in J^{-1}(\mu)$.

On the other hand, if $x \in J^{-1}(\mu)$ and $a_x \in (J^T)^{-1}(0, \mu)$ then, using Theorem 3.11, the fact that $(\tilde{\pi}_\mu, \pi_\mu)$ is a Lie algebroid epimorphism and that $(J^T)^{-1}(0, \mu) \rightarrow J^{-1}(\mu)$ is a Lie subalgebroid of A , one deduces that

$$\begin{aligned} \Omega_\mu(\tilde{\pi}_\mu(\mathcal{H}_f^\Omega(x)), \tilde{\pi}_\mu(a_x)) &= \Omega(\mathcal{H}_f^\Omega(x), a_x) = (d^A f)(a_x) \\ &= (d^{(J^T)^{-1}(0, \mu)}(\tilde{f} \circ \pi_\mu))(a_x) = (d^{A_\mu} \tilde{f})(\tilde{\pi}_\mu(a_x)) \\ &= \Omega_\mu(\mathcal{H}_{\tilde{f}}^{\Omega_\mu}(\pi_\mu(x)), \tilde{\pi}_\mu(a_x)) \end{aligned}$$

So, since Ω_μ is non-degenerate

$$\tilde{\pi}_\mu(\mathcal{H}_f^\Omega(x)) = \mathcal{H}_{\tilde{f}}^{\Omega_\mu}(\pi_\mu(x)).$$

Thus, using again that $(\tilde{\pi}_\mu, \pi_\mu)$ is an epimorphism of Lie algebroids, we conclude that

$$T_x \pi_\mu(\rho_{(J^T)^{-1}(0, \mu)}(\mathcal{H}_f^\Omega(x))) = \rho_{A_\mu}(\mathcal{H}_{\tilde{f}}^{\Omega_\mu}(\pi_\mu(x))), \quad \forall x \in J^{-1}(\mu).$$

Therefore, if $x \in J^{-1}(\mu)$

$$\begin{aligned} \{\tilde{f}, \tilde{g}\}_\mu(\pi_\mu(x)) &= -(\rho_{A_\mu}(\mathcal{H}_{\tilde{f}}^{\Omega_\mu}(\pi_\mu(x))))\tilde{g} \\ &= -(T_x \pi_\mu(\rho_{(J^T)^{-1}(0, \mu)}(\mathcal{H}_f^\Omega(x))))\tilde{g} \\ &= -(\rho_{(J^T)^{-1}(0, \mu)}(\mathcal{H}_f^\Omega(x)))(\tilde{g} \circ \pi_\mu) \\ &= -(\rho(\mathcal{H}_f^\Omega(x))g) = \{f, g\}(x). \end{aligned}$$

□

4. THE CANONICAL COVER OF A FIBERWISE LINEAR POISSON STRUCTURE

A standard example of a symplectic manifold is the cotangent bundle T^*M of a manifold M with its canonical symplectic structure. In the setting of Lie algebroids, the tangent bundle $\pi_{T^*M} : T(T^*M) \rightarrow T^*M$ of T^*M is a symplectic-like Lie algebroid. This symplectic-like Lie algebroid may be considered as the canonical cover of the canonical symplectic structure on T^*M . In fact, it is a particular case of a type of symplectic-like Lie algebroids, *the prolongation of a Lie algebroid A on its dual A^** in the terminology of [13], which may be considered as *the canonical cover of the fiberwise linear Poisson structure of A^** .

In this section we will describe this last canonical cover and will prove that if a Lie group G acts freely and properly on A by complete lifts then one may introduce an equivariant momentum map with respect to a certain canonical action by complete lifts on its canonical cover.

The vector bundle $\mathcal{T}^A A^* \rightarrow A^*$. Let $(A, [\cdot, \cdot], \rho)$ be a Lie algebroid of rank n over a manifold M of dimension m with $\tau : A \rightarrow M$ the associated vector bundle projection and let $F : M' \rightarrow M$ be a smooth map from a manifold M' to M . If $x' \in M'$, we consider the vector subspace

$$(\mathcal{T}^A M')_{x'} = \{(a, v) \in A_{F(x')} \times T_{x'} M' / \rho(a) = T_{x'} F(v)\}$$

of $A_{F(x')} \times T_{x'} M'$ of dimension $n + m' - \dim(\rho(A_{F(x')}) + T_{x'} F(T_{x'} M'))$, where m' is the dimension of M' . If we suppose that $\dim(\rho(A_{F(x')}) + T_{x'} F(T_{x'} M'))$ is constant over $F(M')$ (for instance, if F is a submersion) then $\mathcal{T}^A M'$ is a vector bundle over M' which is called *the prolongation of A over F* (see [9, 13]). In this case, a section \mathcal{X} of $\mathcal{T}^A M' \rightarrow M'$ is said to be *projectable* if there exist a section X of A and a vector field V on M' , F -projectable over $\rho(X)$, satisfying

$$\mathcal{Z}(m') = (X(F(m')), V(m')), \text{ for all } m' \in M'.$$

The section \mathcal{Z} will be denoted by $\mathcal{Z} = (X, V)$. Note that one may choose a local basis $\{\mathcal{Z}_I\}$ of $\Gamma(\mathcal{T}^A A^*)$ such that, for all I , \mathcal{Z}_I is a projectable section.

On the other hand, a section $\tilde{\gamma}$ of the dual vector bundle $(\mathcal{T}^A M')^* \rightarrow M'$ is said to be *projectable* if there exist a section α of A^* and a 1-form β of M' such that

$$\tilde{\gamma}(X, V) = \alpha(X) \circ F + \beta(V), \quad \text{for } (X, V) \text{ a projectable section of } \mathcal{T}^A M'.$$

In such a case we will use $\tilde{\gamma} = (\alpha, \beta)$. Note that one may choose a local basis $\{\mathcal{Z}^I\}$ of $\Gamma((\mathcal{T}^A A^*)^*)$ such that, for all I , \mathcal{Z}^I is a projectable section.

A particular case is when the function F is the dual bundle projection $\tau_* : A^* \rightarrow M$ of the Lie algebroid A . Then the prolongation $\tau_{\mathcal{T}^A A^*} : \mathcal{T}^A A^* \rightarrow A^*$ of A over $\tau_* : A^* \rightarrow M$ is called the *A-tangent bundle of A^** . In such a case, $\dim(\rho(A_x) + T_{\alpha_x} \tau_*(T_{\alpha_x} A^*))$ is just the dimension of M , for all $\alpha_x \in A_x^*$, and the rank of $\mathcal{T}^A A^*$ is $2n$.

A basis of local sections of the vector bundle $\tau_{\mathcal{T}^A A^*} : \mathcal{T}^A A^* \rightarrow A^*$ is defined as follows. If (x^i) are local coordinates on an open subset U of M , $\{e_I\}$ is a basis of sections of the vector bundle $\tau^{-1}(U) \rightarrow U$ and (x^i, y_I) are the corresponding local coordinates on A^* then $\{\mathcal{X}_I, \mathcal{Y}^I\}$ is a local basis of $\Gamma(\mathcal{T}^A A^*)$, where \mathcal{X}_I and \mathcal{Y}^I are the projectable sections defined by

$$(4.25) \quad \mathcal{X}_I = (e_I, \rho_I^i \frac{\partial}{\partial x^i}), \quad \mathcal{Y}^I = (0, \frac{\partial}{\partial y_I}).$$

The symplectic-like Lie algebroid structure on $\mathcal{T}^A A^* \rightarrow A^*$. The vector bundle $\mathcal{T}^A A^*$ admits a Lie algebroid structure $([\![\cdot, \cdot]\!]_{\mathcal{T}^A A^*}, \rho_{\mathcal{T}^A A^*})$ which is characterized by the following conditions

$$[\![(X, V), (X', V')]\!]_{\mathcal{T}^A A^*} = ([\![X, Y]\!], [V, V']), \quad \rho_{\mathcal{T}^A A^*}(X, V) = V,$$

for $(X, V), (X', V')$ projectable sections of $\mathcal{T}^A A^*$.

If $d^{\mathcal{T}^A A^*}$ is the differential associated with this Lie algebroid structure, then

$$(4.26) \quad \begin{aligned} d^{\mathcal{T}^A A^*} f(X_1, V_1) &= df(V_1) \\ d^{\mathcal{T}^A A^*}(\alpha, \beta)((X_1, V_1), (X_2, V_2)) &= d^A \alpha(X_1, X_2) \circ \tau_* + d\beta(V_1, V_2) \end{aligned}$$

where $f : A^* \rightarrow \mathbb{R}$ is a smooth function, $(\alpha, \beta) \in \Gamma((\mathcal{T}^A A^*)^*)$ and $(X_i, V_i) \in \Gamma(\mathcal{T}^A A^*)$ are projectable sections of $(\mathcal{T}^A A^*)^*$ and $\mathcal{T}^A A^*$, respectively.

The canonical section λ_A of the dual bundle to $\mathcal{T}^A A^*$ (which is called the *Liouville section associated with the Lie algebroid A*) may be defined as follows

$$(4.27) \quad \lambda_A(a, v_\alpha) = \alpha(a), \quad \text{for } \alpha \in A^* \text{ and } (a, v_\alpha) \in \mathcal{T}_\alpha^A A^*.$$

The section Ω_A of $\wedge^2(\mathcal{T}^A A^*)^* \rightarrow A^*$ given by

$$\Omega_A = -d^{\mathcal{T}^A A^*} \lambda_A$$

is nondegenerate and $d^{\mathcal{T}^A A^*} \Omega_A = 0$. Thus, Ω_A is a symplectic-like section of the Lie algebroid $\mathcal{T}^A A^* \rightarrow A^*$ which is called the *canonical symplectic-like section associated with the Lie algebroid A* . The Poisson structure on the base space A^* induced by this symplectic-like section is just the linear Poisson structure on A^* associated with the Lie algebroid A (see [13]). For this reason, $\mathcal{T}^A A^*$ may be considered as the *canonical cover of the fiberwise linear Poisson*

structure on A^* . If $\{\mathcal{X}_I, \mathcal{Y}^I\}$ is the local basis of sections of $\mathcal{T}^A A^*$ described in (4.25), the local expressions of λ_A and Ω_A are

$$\lambda_A = y_I \mathcal{X}^I, \quad \Omega_A = \mathcal{X}^I \wedge \mathcal{Y}_I + \frac{1}{2} C_{IJ}^K y_K \mathcal{X}^I \wedge \mathcal{X}^J$$

where $\{\mathcal{X}^I, \mathcal{Y}_I\}$ is the dual basis of $\{\mathcal{X}_I, \mathcal{Y}^I\}$ and C_{IJ}^K are the local structure functions of the bracket $[[\cdot, \cdot]]$ (for more details, see [13]).

Examples 4.1. (i) Note that if A is the standard Lie algebroid TM then the symplectic-like Lie algebroid $\mathcal{T}^A A^* \rightarrow A^*$ may be identified with the standard Lie algebroid $T(T^*M) \rightarrow T^*M$ and, under this identification, Ω_A is the canonical symplectic Ω_M structure of T^*M .

(ii) For the case $A = \mathfrak{g} \times TM$ from (ii) in Examples 3.2, we have that $\mathcal{T}^A A^* \rightarrow A^*$ can be identified with $\mathcal{T}^{\mathfrak{g}} \mathfrak{g}^* \oplus \mathcal{T}^{TM}(T^*M) \rightarrow \mathfrak{g}^* \times T^*M$, i.e.

$$(\mathfrak{g} \times T\mathfrak{g}^*) \oplus T(T^*M) \rightarrow \mathfrak{g}^* \times T^*M.$$

Under this identification, the symplectic-like structure $\Omega_{\mathfrak{g} \times TM}$ on $\mathcal{T}^{\mathfrak{g} \times TM}(\mathfrak{g}^* \times T^*M)$ is just $\Omega_{\mathfrak{g}} \oplus \Omega_M$, where Ω_M is the standard symplectic 2-form on T^*M and $\Omega_{\mathfrak{g}}$ is the symplectic-like structure on $\mathfrak{g} \times T\mathfrak{g}^* \rightarrow \mathfrak{g}^*$ characterized by

$$(\Omega_{\mathfrak{g}})_{\eta_0}((\xi, \eta), (\xi', \eta')) = \eta'(\xi) - \eta(\xi') + \eta_0[\xi, \xi']_{\mathfrak{g}},$$

for all $\eta_0, \eta, \eta' \in \mathfrak{g}^*$ and $\xi, \xi' \in \mathfrak{g}$.

The action of a Lie group G on $\mathcal{T}^A A^* \rightarrow A^*$ by complete lifts. Now, suppose that $\Phi : G \times A \rightarrow A$ is a free and proper action of a connected Lie group G by complete lifts with respect to the Lie algebra anti-morphism $\psi : \mathfrak{g} \rightarrow \Gamma(A)$. Denote by $\phi : G \times M \rightarrow M$ the corresponding action on M and by $\Phi^* : G \times A^* \rightarrow A^*$ the left dual action on A^* . In what follows, we will describe a free and proper canonical action by complete lifts on $\mathcal{T}^A A^*$ induced by Φ .

Proposition 4.2. *Let $\Phi : G \times A \rightarrow A$ be a free and proper action of a connected Lie group G on the Lie algebroid A by complete lifts with respect to $\psi : \mathfrak{g} \rightarrow \Gamma(A)$. Then the map $(\Phi, T\Phi^*) : G \times \mathcal{T}^A A^* \rightarrow \mathcal{T}^A A^*$ given by*

$$(4.28) \quad (\Phi, T\Phi^*)(g, (a_x, v_{\alpha_x})) = (\Phi_g(a_x), (T_{\alpha_x} \Phi_g^*)(v_{\alpha_x})), \quad \alpha_x \in A_x^* \text{ and } (a_x, v_{\alpha_x}) \in \mathcal{T}_{\alpha_x}^A A^*$$

defines a free and proper left canonical action of G on the symplectic-like Lie algebroid $\mathcal{T}^A A^$ by complete lifts with respect to the Lie algebra anti-morphism $\psi^T : \mathfrak{g} \rightarrow \Gamma(\mathcal{T}^A A^*)$ defined by*

$$(4.29) \quad \psi^T(\xi) = (\psi(\xi), \xi_{A^*}), \quad \text{for } \xi \in \mathfrak{g},$$

where ξ_{A^} is the infinitesimal generator of ξ with respect to Φ^* .*

Proof. Note that the map $(\Phi, T\Phi^*)$ is well-defined. In fact, since Φ is an action by complete lifts, then $\Phi_g : A \rightarrow A$ is a Lie algebroid isomorphism, for all $g \in G$. So, using (2.5) with $k = 0$, we have that

$$(4.30) \quad \rho(\Phi_g(a_x)) = T_x \phi_g(\rho(a_x)), \quad \text{for all } g \in G, x \in M \text{ and } a_x \in A_x.$$

Moreover,

$$(4.31) \quad T_{\Phi_g^*(\alpha_x)} \tau_*(T_{\alpha_x} \Phi_g^*(v_{\alpha_x})) = T_{\alpha_x}(\phi_g \circ \tau_*)(v_{\alpha_x}) = T_x \phi_g(\rho(a_x)),$$

for all $v_{\alpha_x} \in T_{\alpha_x} A^*$. Thus, from (4.30) and (4.31), we deduce that

$$T_{\Phi_g^*(\alpha_x)} \tau_*(T_{\alpha_x} \Phi_g^*(v_{\alpha_x})) = \rho(\Phi_g(a_x))$$

for all $(a_x, v_{\alpha_x}) \in \mathcal{T}_{\alpha_x}^A A^*$, that is,

$$(\Phi, T^* \Phi)(g, (a_x, v_{\alpha_x})) \in \mathcal{T}_{\Phi_g^*(\alpha_x)}^A A^*.$$

Obviously, $(\Phi, T^* \Phi)$ is a free and proper action. We will now show that this action on $\mathcal{T}^A A^*$ is by complete lifts. Firstly, note that the map $\psi^T : \mathfrak{g} \rightarrow \Gamma(\mathcal{T}^A A^*)$ is well defined. In fact, since $\rho(\psi(\xi))$ is just the infinitesimal generator ξ_M of ξ with respect to the action $\phi : G \times M \rightarrow M$ and the projection $\tau_* : A^* \rightarrow M$ is equivariant, we have that

$$\rho(\psi(\xi)) = T\tau_*(\xi_{A^*}), \text{ for all } \xi \in \mathfrak{g}.$$

On the other hand, the infinitesimal generator $\xi_{\mathcal{T}^A A^*} \in \mathfrak{X}(\mathcal{T}^A A^*)$ of $\xi \in \mathfrak{g}$ with respect to the action $(\Phi, T^* \Phi)$ is the pair $(\xi_A, \xi_{A^*}^c)$, where ξ_A is the infinitesimal generator of $\xi \in \mathfrak{g}$ with respect to Φ and $\xi_{A^*}^c$ is the complete lift of ξ_{A^*} . Moreover, the complete lift of $\psi^T(\xi)$ with respect to the Lie algebroid $\mathcal{T}^A A^*$ is just $(\psi(\xi)^c, \xi_{A^*}^c)$. This is a consequence of the fact that $(\psi(\xi)^c, \xi_{A^*}^c) \in \mathfrak{X}(\mathcal{T}^A A^*)$ is $\tau_{\mathcal{T}^A A^*}$ -projectable on $\rho_{\mathcal{T}^A A^*}(\psi(\xi), \xi_{A^*}) = \xi_{A^*}$ and that, from (4.26), we deduce that

$$\mathcal{L}_{(\psi(\xi), \xi_{A^*})}^{\mathcal{T}^A A^*}(\alpha, \beta) = (\mathcal{L}_{\psi(\xi)}^A \alpha, \mathcal{L}_{\xi_{A^*}} \beta)$$

for every projectable section (α, β) on $\Gamma((\mathcal{T}^A A^*)^*)$. Here \mathcal{L} is the standard Lie derivative.

Therefore, $(\Phi, T^* \Phi)$ is an action by complete lifts and consequently by automorphisms of Lie algebroids. Finally, a direct computation, using (4.27), proves that the action $(\Phi, T^* \Phi)$ preserves the Liouville section λ_A , i.e

$$(\Phi, T^* \Phi)_g^* \lambda_A = \lambda_A, \text{ for all } g \in G.$$

Thus, using (2.5) and the fact that $(\Phi, T^* \Phi)_g$ is an automorphism of Lie algebroids, we conclude that $(\Phi, T^* \Phi)_g$ preserves the canonical symplectic-like section Ω_A of $\mathcal{T}^A A^*$. \square

The momentum map for the canonical action of G on the Lie algebroid $\mathcal{T}^A A^* \rightarrow A^*$.

Denote by $J_{A^*} : A^* \rightarrow \mathfrak{g}^*$ the map given by

$$(4.32) \quad J_{A^*}(\alpha_x)(\xi) = \alpha_x(\psi(\xi)(x)), \quad \text{with } x \in M, \alpha_x \in A_x^* \text{ and } \xi \in \mathfrak{g}.$$

Then, we have the following result

Proposition 4.3. *The map $J_{A^*} : A^* \rightarrow \mathfrak{g}^*$ is an equivariant momentum map for the Poisson action $\Phi^* : G \times A^* \rightarrow A^*$.*

Proof. From Proposition 3.1 we have that if Π_{A^*} is the linear Poisson structure on A^* and $\widehat{\psi(\xi)}$ is the linear function associated with the section $\psi(\xi) \in \Gamma(A)$, for each $\xi \in \mathfrak{g}$, the Hamiltonian vector field

$$H_{\widehat{\psi(\xi)}}^{\Pi_{A^*}} = i_{d\widehat{\psi(\xi)}} \Pi_{A^*} \in \mathfrak{X}(A^*)$$

is just the infinitesimal generator $\xi_{A^*} \in \mathfrak{X}(A^*)$ of ξ with respect to the action Φ^* . Note that the function $(J_{A^*})_\xi : A^* \rightarrow \mathbb{R}$ given by

$$(J_{A^*})_\xi(\alpha_x) = (J_{A^*}(\alpha_x))(\xi)$$

is just $\widehat{\psi(\xi)}$. Thus, J_{A^*} is a momentum map for the Poisson action $\Phi^* : G \times A^* \rightarrow A^*$. Now, we will prove that J_{A^*} is equivariant, i.e.,

$$J_{A^*} \circ \Phi_g^* = \text{Coad}_g^G \circ J_{A^*}.$$

Indeed, if $x \in M$ and $\alpha_x \in A^*$, then, from Proposition 3.4, we have that

$$\begin{aligned} J_{A^*}(\Phi_g^*(\alpha_x))(\xi) &= (\Phi_g^*(\alpha_x))(\psi(\xi)(\phi_g(x))) = \alpha_x(\Phi_{g^{-1}}(\psi(\xi)(\phi_g(x)))) \\ &= \alpha_x(\psi(\text{Ad}_{g^{-1}}^G \xi)(x)) = J_{A^*}(\alpha_x)(\text{Ad}_{g^{-1}}^G \xi) \\ &= ((\text{Coad}_g^G)(J_{A^*}(\alpha_x)))(\xi), \end{aligned}$$

for all $\xi \in \mathfrak{g}$. □

Now, using Lemma 3.8, we have that the map

$$(4.33) \quad J_{A^*}^T : \mathcal{T}^A A^* \rightarrow \mathfrak{g}^* \times \mathfrak{g}^*, \quad J_{A^*}^T(a_x, v_{\alpha_x}) = (T_{\alpha_x} J_{A^*}(v_{\alpha_x}), J_{A^*}(\alpha_x))$$

is equivariant with respect to the action $(\Phi, T\Phi^*)^T : TG \times \mathcal{T}^A A^* \rightarrow \mathcal{T}^A A^*$.

From the injectivity of ψ_x (see Remark 3.3), it follows that the restriction of $J_{A^*} : A^* \rightarrow \mathfrak{g}^*$ to A_x^* is a linear epimorphism and therefore, for all $\alpha_x \in A_x^*$ the restriction of the tangent map $T_{\alpha_x} J_{A^*} : T_{\alpha_x} A^* \rightarrow T_{J_{A^*}(\alpha_x)} \mathfrak{g}^* \cong \mathfrak{g}^*$ to $T_{\alpha_x} A_x^*$ is surjective. Thus, all the elements of \mathfrak{g}^* are regular values of J_{A^*} and

$$T_{\alpha_x} J_{A^*} \circ (\rho_{\mathcal{T}^A A^*})_{\alpha_x} : \mathcal{T}_{\alpha_x}^A A^* \rightarrow \mathfrak{g}^*$$

is surjective, for all $\alpha_x \in A_x^*$. Note that

$$T_{\alpha_x} A_x^* = \ker T_{\alpha_x} \tau_* \subseteq (\rho_{\mathcal{T}^A A^*})_{\alpha_x}(\mathcal{T}_{\alpha_x}^A A^*).$$

In conclusion if $\mu \in \mathfrak{g}^*$, then $J_{A^*}^{-1}(\mu)$ is a regular submanifold of A^* and $(J_{A^*}^T)^{-1}(0, \mu)$ is a Lie subalgebroid of $\mathcal{T}^A A^*$ over $J_{A^*}^{-1}(\mu)$ (see Proposition 3.9). In fact, $(J_{A^*}^T)^{-1}(0, \mu)$ is just the prolongation

$$\mathcal{T}^A(J_{A^*}^{-1}(\mu))$$

of the Lie algebroid A over the restriction $(\tau_*)|_{(J_{A^*})^{-1}(\mu)} : J_{A^*}^{-1}(\mu) \rightarrow M$ of $\tau_* : A^* \rightarrow M$ to the submanifold $J_{A^*}^{-1}(\mu)$. Note that $J_{A^*}^{-1}(\mu)$ is an affine subbundle of A^* over M and that $(\tau_*)|_{J_{A^*}^{-1}(\mu)} : J_{A^*}^{-1}(\mu) \rightarrow M$ is the projection.

5. THE REDUCTION OF THE CANONICAL COVER OF A FIBERWISE LINEAR POISSON STRUCTURE

Let $(A, [\cdot, \cdot], \rho)$ be a Lie algebroid over the manifold M and $\tau : A \rightarrow M$ the vector bundle projection. Suppose that $\Phi : G \times A \rightarrow A$ is a free and proper action of a connected Lie group G by complete lifts with respect to the Lie algebra anti-morphism $\psi : \mathfrak{g} \rightarrow A$. In the previous sections, we have shown that in this situation, we have a free and proper canonical action $(\Phi, T\Phi^*) : G \times \mathcal{T}^A A^* \rightarrow \mathcal{T}^A A^*$ of the Lie group G on the symplectic-like Lie algebroid $\mathcal{T}^A A^*$ by complete lifts with respect to the Lie algebra anti-morphism $\psi^T : \mathfrak{g} \rightarrow \Gamma(\mathcal{T}^A A^*)$ given in

(4.29). In addition, we have an equivariant momentum map $J_{A^*} : A^* \rightarrow \mathfrak{g}^*$ on A^* with respect to the left Poisson action $\Phi^* : G \times A^* \rightarrow A^*$ of G on A^* .

If μ is an element of \mathfrak{g}^* then we obtain, in a natural, way a free and proper action $(\Phi, T\Phi^*) : G_\mu \times \mathcal{T}^A J_{A^*}^{-1}(\mu) \rightarrow \mathcal{T}^A J_{A^*}^{-1}(\mu)$ of the isotropy group of μ on the Lie algebroid $\mathcal{T}^A J_{A^*}^{-1}(\mu)$ by restriction. Now, using Theorem 3.11, we conclude that the reduced vector bundle

$$(\mathcal{T}^A A^*)_\mu = \mathcal{T}^A J_{A^*}^{-1}(\mu)/TG_\mu \rightarrow J_{A^*}^{-1}(\mu)/G_\mu$$

is a symplectic-like Lie algebroid with symplectic-like section Ω_μ characterized by

$$\tilde{\pi}_\mu^* \Omega_\mu = \tilde{\iota}_\mu^* \Omega_A$$

where $\tilde{\pi}_\mu : \mathcal{T}^A J_{A^*}^{-1}(\mu) \rightarrow (\mathcal{T}^A A^*)_\mu$ is the canonical projection, $\tilde{\iota}_\mu : \mathcal{T}^A J_{A^*}^{-1}(\mu) \rightarrow \mathcal{T}^A A^*$ is the inclusion and Ω_A is the standard symplectic-like structure on $\mathcal{T}^A A^*$.

In what follows, we will describe this reduced Lie algebroid $(\mathcal{T}^A A^*)_\mu$. Firstly, we will discuss the case $\mu = 0$.

5.1. The case $\mu = 0$. Note that, under this assumption, the isotropy group G_μ is just G . We will prove that the reduced symplectic-like Lie algebroid $(\mathcal{T}^A A^*)_0 = \mathcal{T}^A J_{A^*}^{-1}(0)/TG \rightarrow J_{A^*}^{-1}(0)/G$ is the canonical cover of a fiberwise linear Poisson structure on the dual A_0^* of a certain Lie algebroid A_0 over M/G .

Description of the Lie algebroid A_0 . The Lie algebroid A_0 over M/G is the space of orbits A/TG of the affine action of TG on A (see Theorem 3.6). As we know (see the proof of Theorem 3.6), if $\tilde{\pi} : A \rightarrow A_0 = A/TG$ and $\pi : M \rightarrow M/G$ are the canonical projections and $(\llbracket \cdot, \cdot \rrbracket_{A_0}, \rho_{A_0})$ is the Lie algebroid structure on A_0 then

$$(5.34) \quad \llbracket X_0, Y_0 \rrbracket_{A_0} \circ \pi = \tilde{\pi}(\llbracket X, Y \rrbracket), \quad \rho_{A_0}(X_0) = T\pi(\rho(X))$$

for $X_0, Y_0 \in \Gamma(A_0)$ and $X, Y \in \Gamma(A)$ satisfying

$$X_0 \circ \pi = \tilde{\pi} \circ X, \quad Y_0 \circ \pi = \tilde{\pi} \circ Y.$$

Note that with this structure, $\tilde{\pi} : A \rightarrow A_0$ is an epimorphism of Lie algebroids.

Now, we will prove that the vector bundle $A_0 = A/TG \rightarrow M/G$ is isomorphic to

$$(J_{A^*}^{-1}(0)/G)^* \rightarrow M/G.$$

In fact, one may easily test that the submanifold $J_{A^*}^{-1}(0)$ is just the annihilator $(\psi(\mathfrak{g}))^0$ of $\psi(\mathfrak{g})$. Therefore, the restriction $\tau_*^0 = \tau_{*|J_{A^*}^{-1}(0)} : J_{A^*}^{-1}(0) \rightarrow M$ of $\tau_* : A^* \rightarrow M$ to this submanifold is a vector bundle over M . Moreover, a direct computation proves that this vector bundle is isomorphic to the dual vector bundle $(A/\psi(\mathfrak{g}))^*$ of $A/\psi(\mathfrak{g}) \rightarrow M$.

Therefore, using the equivalences (3.13), we deduce that the three vector bundles

$$A_0 = A/TG \rightarrow M/G, \quad (A/\psi(\mathfrak{g}))/G \rightarrow M/G, \quad (J_{A^*}^{-1}(0)/G)^* \cong (J_{A^*}^{-1}(0))^*/G \rightarrow M/G$$

are isomorphic. Thus, we may induce isomorphic Lie algebroid structures on these vector bundles.

The description of the Lie algebroid isomorphism between $(\mathcal{T}^A A^)_0$ and $\mathcal{T}^{A_0} A_0^*$.* In what follows we identify $A_0 = A/TG$ with $(J_{A^*}^{-1}(0)/G)^* \cong (J_{A^*}^{-1}(0))^*/G$. Under this identification, we denote

by $\varphi : A \rightarrow (J_{A^*}^{-1}(0)/G)^*$ the epimorphism of vector bundles corresponding to the quotient projection $A \rightarrow A/TG$.

Let us consider the following epimorphism of vector bundles over $\pi_0 : J_{A^*}^{-1}(0) \rightarrow A_0^* = J_{A^*}^{-1}(0)/G$

$$(5.35) \quad \varphi^T : \mathcal{T}^A J_{A^*}^{-1}(0) \rightarrow \mathcal{T}^{A_0} A_0^*, \quad (a_x, v_{\alpha_x}) \mapsto (\varphi(a_x), T_{\alpha_x} \pi_0(v_{\alpha_x})).$$

Note that, using (5.34) and the fact that $\tilde{\tau}_*^0 \circ \pi_0 = \pi \circ \tau_*^0$, we have that φ^T is well-defined. Here $\tilde{\tau}_*^0 : J_{A^*}^{-1}(0)/G \rightarrow M/G$ is the dual vector bundle of $(J_{A^*}^{-1}(0)/G)^* \rightarrow M/G$.

Now, since that $\varphi : A \rightarrow A_0$ is a Lie algebroid epimorphism, then φ^T is also a Lie algebroid epimorphism. Furthermore, it is easy to prove, using that φ is TG -invariant, that φ^T is TG -invariant with respect to the action $(\Phi, T\Phi^*)^T$ of TG restricted to $\mathcal{T}^A J_{A^*}^{-1}(0)$. In fact,

$$\begin{aligned} \varphi^T((\Phi, T\Phi^*)_{(g, \xi)}^T(a_x, v_{\alpha_x})) &= (\varphi(\Phi_g(a_x)) + \varphi(\Phi_g(\psi(\xi))), T_{\alpha_x} \pi_0(T_x \Phi_g(v_{\alpha_x} + \xi_{A^*}))) \\ &= (\varphi(a_x), T_{\alpha_x} \pi_0(v_{\alpha_x})) = \varphi^T(a_x, v_{\alpha_x}) \end{aligned}$$

for all $(g, \xi) \in G \times \mathfrak{g} \cong TG$ and $(a_x, v_{\alpha_x}) \in \mathcal{T}_{\alpha_x}^A J_{A^*}^{-1}(0)$. Note that

$$\varphi(\psi(\xi)) = \varphi(\Phi_{(e, \xi)}(0)) = 0.$$

Thus, we have the following Lie algebroid epimorphism $\bar{\varphi}^T$ between $(\mathcal{T}^A A^*)_0$ and $\mathcal{T}^{A_0} A_0^*$ over the identity of A_0^*

$$(5.36) \quad \begin{array}{ccc} (\mathcal{T}^A A^*)_0 & \xrightarrow{\bar{\varphi}^T} & \mathcal{T}^{A_0} A_0^* \\ \tau_0^T \downarrow & & \downarrow \tau_{\mathcal{T}^{A_0} A_0^*} \\ A_0^* & \xrightarrow{Id} & A_0^* \end{array}$$

In fact,

$$(5.37) \quad \bar{\varphi}^T[(a_x, v_{\alpha_x})] = (\varphi(a_x), (T_{\alpha_x} \pi_0)(v_{\alpha_x})), \text{ for } (a_x, v_{\alpha_x}) \in \mathcal{T}_{\alpha_x}^A J_{A^*}^{-1}(0).$$

Finally, we will prove that $\bar{\varphi}^T$ is an isomorphism, that is, $\bar{\varphi}^T$ is injective.

If $(e_x, v_{\alpha_x}) \in \ker \bar{\varphi}^T$ then $e_x \in \ker \varphi = \psi_x(\mathfrak{g})$ and v_{α_x} is a vertical vector with respect to $\pi_0 : (J_{A^*})^{-1}(0) \rightarrow (J_{A^*})^{-1}(0)/G$. Then, there are $\xi, \xi' \in \mathfrak{g}$ such that

$$e_x = \psi_x(\xi) \text{ and } v_{\alpha_x} = \xi'_{A^*}(\alpha_x).$$

Then,

$$\xi_M(x) = \rho(e_x) = T_{\alpha_x} \tau_*^0(v_{\alpha_x}) = \xi'_M(x)$$

and, since $\phi : G \times M \rightarrow M$ is a free action, we conclude that $\xi = \xi'$. Therefore,

$$(e_x, v_{\alpha_x}) = (\psi_x(\xi), \xi_{A^*}(\alpha_x)) = (\Phi, T\Phi^*)^T((e, \xi), (0, 0))$$

where e is the identity element of G . Thus, $\bar{\varphi}^T$ is injective.

The Lie algebroid isomorphism between $(\mathcal{T}^A A^*)_0$ and $\mathcal{T}^{A_0} A_0^*$ is canonical. We will see that $\bar{\varphi}^T$ is canonical, i.e.

$$(5.38) \quad (\bar{\varphi}^T)^* \Omega_{A_0} = \Omega_0,$$

where Ω_0 is the reduced symplectic-like structure on $(\mathcal{T}^A A^*)_0$ given in Theorem 3.11 and Ω_{A_0} is the canonical symplectic-like structure on $\mathcal{T}^{A_0} A_0^*$.

From (4.27), (5.35) and the definition of the morphism $\varphi : A \rightarrow A_0$, we obtain that

$$(\varphi^T)^* \lambda_{A_0} = \tilde{\iota}_0^* \lambda_A$$

where λ_{A_0} (respectively, λ_A) is the Liouville section of $A_0 \rightarrow M/G$ (respectively, of $A \rightarrow M$) and $\tilde{\iota}_0 : \mathcal{T}^A J_{A^*}^{-1}(0) \rightarrow \mathcal{T}^A A^*$ is the inclusion.

Thus, since φ^T and $\tilde{\iota}_0$ are Lie algebroid morphisms, we obtain that

$$(\varphi^T)^* \Omega_{A_0} = \tilde{\iota}_0^* \Omega_A.$$

On the other hand, if $\tilde{\pi}_0 : \mathcal{T}^A J_{A^*}^{-1}(0) \rightarrow (\mathcal{T}^A A^*)_0$ is the canonical projection, it is clear that $\tilde{\varphi}^T \circ \tilde{\pi}_0 = \varphi^T$ which implies that

$$\tilde{\pi}_0^* ((\tilde{\varphi}^T)^* \Omega_{A_0}) = \tilde{\pi}_0^* \Omega_0$$

and, therefore,

$$(5.39) \quad (\tilde{\varphi}^T)^* \Omega_{A_0} = \Omega_0.$$

In the following theorem we summarize the results obtained in the case $\mu = 0$.

Theorem 5.1. *Let $(A, \llbracket \cdot, \cdot \rrbracket, \rho)$ be a Lie algebroid on the manifold M and $\Phi : G \times A \rightarrow A$ a free and proper action of a connected Lie group by complete lifts. Then, the reduced symplectic-like Lie algebroid*

$$(\mathcal{T}^A A^*)_0 = (\mathcal{T}^A J_{A^*}^{-1}(0))/TG \rightarrow J_{A^*}^{-1}(0)/G$$

is canonically isomorphic to the Lie algebroid $\mathcal{T}^{A_0} A_0^$, equipped with the standard symplectic-like structure, where the Lie algebroid A_0 is the vector bundle*

$$A_0 = A/TG \rightarrow M/G$$

endowed with the quotient Lie algebroid structure characterized by (5.34).

5.2. The case $\mathbf{G}_\mu = \mathbf{G}$. Suppose that the assumptions of Theorem 5.1 hold. Additionally, we consider a principal G -connection $\mathcal{A} : TM \rightarrow \mathfrak{g}$ for the corresponding principal bundle $\pi : M \rightarrow M/G$. In such a case we have a vector bundle morphism $\mathcal{A}^A : A \rightarrow \mathfrak{g}$ given by

$$\mathcal{A}^A(a_x) = \mathcal{A}(\rho_x(a_x)), \quad \forall a_x \in A_x,$$

which satisfies the following properties:

(i) \mathcal{A}^A is equivariant with respect to $\Phi : G \times A \rightarrow A$ and the adjoint action, that is,

$$\mathcal{A}^A(\Phi_g(a_x)) = \text{Ad}_g^G(\mathcal{A}^A(a_x)), \quad \forall a_x \in A_x,$$

(ii) $\mathcal{A}^A(\psi(\xi)(x)) = \xi$, for all $\xi \in \mathfrak{g}$ and $x \in M$.

Note that if $\pi : M \rightarrow M/G$ is the quotient projection then we have

$$TQ = V\pi \oplus H \text{ and } A = \psi(\mathfrak{g}) \oplus H^A,$$

where $V\pi$ is the vertical bundle of π and H^A (respectively, H) is the vector bundle on M whose fiber at $x \in M$ is the vector space

$$H_x^A = \{a_x \in A_x / \mathcal{A}^A(a_x) = 0\} \text{ (respectively, } H_x = \{v_x \in T_x M / \mathcal{A}(v_x) = 0\}).$$

Moreover, H and H^A are G -invariant vector bundles, that is,

$$H_{\phi_g(x)}^A = \Phi_g(H_x^A) \text{ and } H_{\phi_g(x)} = T_x \phi_g(H_x), \quad \forall g \in G.$$

Now, if $\mu \in \mathfrak{g}^*$, we consider the section α_μ of A^* given by

$$\alpha_\mu(a_x) = \mu(\mathcal{A}^A(a_x)),$$

with $x \in M$ and $a_x \in A_x$. This section has the following properties:

(i) $\alpha_\mu(M) \subseteq J_{A^*}^{-1}(\mu)$. In fact,

$$J_{A^*}(\alpha_\mu(x))(\xi) = \alpha_\mu(\psi(\xi)(x)) = \mu(\mathcal{A}^A(\psi(\xi)(x))) = \mu(\xi)$$

for all $x \in M$ and $\xi \in \mathfrak{g}$.

(ii) $\Phi_g^* \alpha_\mu = \alpha_{\text{Coad}_g^* \mu}$, for all $g \in G$, which is a consequence from the equivariance properties of \mathcal{A}^A .

Thus, since $G_\mu = G$ then we deduce that α_μ is G -invariant, i.e.

$$(5.40) \quad \Phi_g^*(\alpha_\mu) = \alpha_\mu.$$

So, in what follows we assume that there is a G -invariant 1-section $\alpha_\mu \in \Gamma(A^*)$ of A^* with values in $J_{A^*}^{-1}(\mu)$. Using (5.40), Proposition 2.2 and the fact that the flow of $\psi(\xi)$ is $\{\Phi_{\exp(t\xi)}\}_{t \in \mathbf{R}}$, we obtain that

$$(5.41) \quad \mathcal{L}_{\psi(\xi)}^A \alpha_\mu = 0.$$

On the other hand,

$$i_{\psi(\xi)} \alpha_\mu = \mu(\mathcal{A}^A(\psi(\xi))) = \mu(\xi).$$

Then

$$(5.42) \quad 0 = \mathcal{L}_{\psi(\xi)}^A \alpha_\mu = i_{\psi(\xi)} d^A \alpha_\mu.$$

Denote by $\beta_\mu = d^A \alpha_\mu$. From (5.40) and since $\Phi_g : A \rightarrow A$ is a Lie algebroid morphism we deduce that the 2-section β_μ of A^* is G -invariant. Moreover, it satisfies $i_{\psi(\xi)} \beta_\mu = 0$ which implies that

$$(\Phi_{(g,\xi)}^T)^* \beta_\mu = \Phi_g^* \beta_\mu = \beta_\mu$$

for all $(g, \xi) \in G \times \mathfrak{g} \cong TG$.

Therefore, there exists a unique $B_\mu \in \Gamma(\wedge^2 A_0^*)$ with the following property:

$$(5.43) \quad \tilde{\pi}^* B_\mu = \beta_\mu = d^A \alpha_\mu,$$

where $\tilde{\pi} : A \rightarrow A_0$ is the corresponding projection. It is clear that $d^{A_0} B_\mu = 0$.

The 2-section B_μ of A_0^* is said to be the *magnetic term associated with* α_μ .

Now, we will prove that there is a Lie algebroid isomorphism, $\Upsilon_{\alpha_\mu} : (\mathcal{T}^A A^*)_\mu \rightarrow \mathcal{T}^{A_0} A_0^*$ between the reduced Lie algebroid $(\mathcal{T}^A A^*)_\mu$ and $\mathcal{T}^{A_0} A_0^*$ such that the symplectic-like section Ω_{A_0} on $\mathcal{T}^{A_0} A_0^*$ and the reduced symplectic-like section Ω_μ on $(\mathcal{T}^A A^*)_\mu$ are related by the following formula

$$\Upsilon_{\alpha_\mu}^* (\Omega_{A_0} - pr_1^* B_\mu) = \Omega_\mu,$$

where $pr_1 : \mathcal{T}^{A_0} A_0^* \rightarrow A_0$ is the Lie algebroid morphism induced by the first projection.

The description of the Lie algebroid isomorphism $\Upsilon_{\alpha_\mu} : (\mathcal{T}^A A^*)_\mu \rightarrow \mathcal{T}^{A_0} A_0^*$. Firstly, we will describe a Lie algebroid morphism between the reduced spaces $(\mathcal{T}^A A^*)_\mu$ and $(\mathcal{T}^A A^*)_0$. Then, we may use Theorem 5.1 in order to construct the isomorphism Υ_{α_μ} .

Using the fact that $\alpha_\mu(M) \subseteq J_{A^*}^{-1}(\mu)$, we deduce that $J_{A^*}^{-1}(\mu) \rightarrow M$ is an affine bundle on M such that

$$J_{A^*}^{-1}(\mu) \cap A_x^* = \{\beta_x \in A_x^* / \beta_x - \alpha_\mu(x) \in J_{A^*}^{-1}(0)\}$$

for all $x \in M$.

Now, we consider the affine bundle isomorphism

$$\begin{array}{ccc} J_{A^*}^{-1}(\mu) & \xrightarrow{sh_\mu} & J_{A^*}^{-1}(0) \\ \downarrow & & \downarrow \\ M & \xrightarrow{Id} & M \end{array}$$

where $sh_\mu(\beta_x) = \beta_x - \alpha_\mu(x)$, for all $\beta_x \in J_{A^*}^{-1}(\mu) \cap A_x^*$.

From the G -invariance of α_μ , we deduce that sh_μ is equivariant with respect to the action $\Phi^* : G \times A^* \rightarrow A^*$, i.e.

$$(sh_\mu \circ \Phi_g^*)(\beta_x) = (\Phi_g^* \circ sh_\mu)(\beta_x),$$

for all $\beta_x \in A_x^* \cap J_{A^*}^{-1}(\mu)$ and $g \in G$. Moreover, one may induce a morphism of vector bundles

$$\begin{array}{ccc} \mathcal{T}^A J_{A^*}^{-1}(\mu) & \xrightarrow{\mathcal{T}^A sh_\mu} & \mathcal{T}^A J_{A^*}^{-1}(0) \\ \downarrow \tau_{\mathcal{T}^A J_{A^*}^{-1}(\mu)} & & \downarrow \tau_{\mathcal{T}^A J_{A^*}^{-1}(0)} \\ J_{A^*}^{-1}(\mu) & \xrightarrow{sh_\mu} & J_{A^*}^{-1}(0) \end{array}$$

where $\mathcal{T}^A sh_\mu(a_x, X_{\beta_x}) = (a_x, T_{\beta_x} sh_\mu(X_{\beta_x}))$, with $(a_x, X_{\beta_x}) \in \mathcal{T}^A J_{A^*}^{-1}(\mu)$. Note that, since

$$\tau_{*|J_{A^*}^{-1}(0)} \circ sh_\mu = \tau_{*|J_{A^*}^{-1}(\mu)},$$

then $\mathcal{T}^A sh_\mu$ is well-defined. Furthermore, a direct proof shows that $\mathcal{T}^A sh_\mu$ is an isomorphism of vector bundles. In fact, one can easily see that $\mathcal{T}^A sh_\mu$ is a Lie algebroid isomorphism, taking into account that

$$\mathcal{T}^A sh_\mu([X, Y], [U, V]) = ([X, Y], [Tsh_\mu \circ U \circ sh_\mu^{-1}, Tsh_\mu \circ V \circ sh_\mu^{-1}]) \circ sh_\mu,$$

$$\rho_{\mathcal{T}^A J_{A^*}^{-1}(0)}((\mathcal{T}^A sh_\mu)(X, U)) = (Tsh_\mu \circ \rho_{\mathcal{T}^A J_{A^*}^{-1}(\mu)})(X, U)$$

for all $X, Y \in \Gamma(A)$ and $U, V \in \mathfrak{X}(J_{A^*}^{-1}(\mu))$ which are $(\tau_*)|_{J_{A^*}^{-1}(\mu)}$ -projectable on $\rho(X)$ and $\rho(Y)$, respectively.

Moreover, since sh_μ is equivariant, we deduce that $\mathcal{T}^A sh_\mu$ is equivariant with respect to the action $(\Phi, T\Phi^*)^T$ of G restricted to $\mathcal{T}^A J_{A^*}^{-1}(\mu)$ and $\mathcal{T}^A J_{A^*}^{-1}(0)$, respectively.

Thus, one induces a Lie algebroid isomorphism

$$\begin{array}{ccc}
(\mathcal{T}^A J_{A^*}^{-1}(\mu))/TG & \xrightarrow{\widetilde{\mathcal{T}^A sh_\mu}} & (\mathcal{T}^A J_{A^*}^{-1}(0))/TG \\
\downarrow \widetilde{\tau}_{\mathcal{T}^A J_{A^*}^{-1}(\mu)} & & \downarrow \widetilde{\tau}_{\mathcal{T}^A J_{A^*}^{-1}(0)} \\
J_{A^*}^{-1}(\mu)/G & \xrightarrow{\widetilde{sh}_\mu} & J_{A^*}^{-1}(0)/G
\end{array}$$

Finally, the isomorphism $\Upsilon_{\alpha_\mu} : (\mathcal{T}^A A^*)_\mu \rightarrow \mathcal{T}^{A_0} A_0^*$ is defined as follows

$$\Upsilon_{\alpha_\mu} = \bar{\varphi}^T \circ \widetilde{\mathcal{T}^A sh_\mu}$$

where $\bar{\varphi}^T : (\mathcal{T}^A A^*)_0 \rightarrow \mathcal{T}^{A_0} A_0^*$ is the Lie algebroid isomorphism defined by (5.37).

Relation between the symplectic-like structures on $(\mathcal{T}^A A^)_\mu$ and $\mathcal{T}^{A_0} A_0^*$.* Let λ_A be the Liouville section on $\mathcal{T}^A A^*$ and $\iota_0 : \mathcal{T}^A (J_{A^*})^{-1}(0) \rightarrow \mathcal{T}^A A^*$ (respectively, $\iota_\mu : \mathcal{T}^A (J_{A^*})^{-1}(\mu) \rightarrow \mathcal{T}^A A^*$) be the corresponding inclusion. Then,

$$((\mathcal{T}^A sh_\mu)^*(\iota_0^* \lambda_A))(a_x, X_{\beta_x}) = (\iota_\mu^* \lambda_A)(a_x, X_{\beta_x}) - \alpha_\mu(a_x),$$

for all $\beta_x \in J_{A^*}^{-1}(\mu)$ and $(a_x, X_{\beta_x}) \in \mathcal{T}_{\beta_x}^A J_{A^*}^{-1}(\mu)$.

On the other hand, if $pr_1^0 : \mathcal{T}^A J_{A^*}^{-1}(0) \rightarrow A$ is the Lie algebroid morphism induced by the first projection, we have that

$$((pr_1^0 \circ \mathcal{T}^A sh_\mu)^* \alpha_\mu)(a_x, X_{\beta_x}) = \alpha_\mu(a_x).$$

This implies that

$$(\mathcal{T}^A sh_\mu)^*(\iota_0^* \lambda_A + (pr_1^0)^* \alpha_\mu) = \iota_\mu^* \lambda_A$$

and thus, from Theorem 3.11, we deduce that

$$(5.44) \quad (\mathcal{T}^A sh_\mu)^*(\tilde{\pi}_0^* \Omega_0 - (pr_1^0)^* \beta_\mu) = \tilde{\pi}_\mu^* \Omega_\mu,$$

where Ω_0 (respectively, Ω_μ) is the symplectic-like structure on $(\mathcal{T}^A J_{A^*}^{-1}(0))/TG$ (respectively, $(\mathcal{T}^A J_{A^*}^{-1}(\mu))/TG$) and $\tilde{\pi}_0 : \mathcal{T}^A J_{A^*}^{-1}(0) \rightarrow (\mathcal{T}^A J_{A^*}^{-1}(0))/TG$ (respectively, $\tilde{\pi}_\mu : \mathcal{T}^A J_{A^*}^{-1}(\mu) \rightarrow (\mathcal{T}^A J_{A^*}^{-1}(\mu))/TG$) is the canonical projection.

Now, using the relations

$$\tilde{\pi}_0 \circ \mathcal{T}^A sh_\mu = \widetilde{\mathcal{T}^A sh_\mu} \circ \tilde{\pi}_\mu \quad \text{and} \quad \tilde{\pi} \circ pr_1^0 \circ \mathcal{T}^A sh_\mu = pr_1 \circ \Upsilon_{\alpha_\mu} \circ \tilde{\pi}_\mu,$$

and the facts

$$(\bar{\varphi}^T)^* \Omega_{A_0} = \Omega_0 \quad \text{and} \quad \tilde{\pi}^* B_\mu = \beta_\mu,$$

we conclude that (5.44) is equivalent to

$$\tilde{\pi}_\mu^*(\Upsilon_{\alpha_\mu}^* \Omega_{A_0} - \Upsilon_{\alpha_\mu}^*(pr_1^*(B_\mu))) = \tilde{\pi}_\mu^* \Omega_\mu.$$

Therefore,

$$\Upsilon_{\alpha_\mu}^*(\Omega_{A_0} - pr_1^*(B_\mu)) = \Omega_\mu.$$

The results obtained in this case may be summarized in the following theorem.

Theorem 5.2. *Let $(A, [\cdot, \cdot], \rho)$ be a Lie algebroid on the manifold M and $\Phi : G \times A \rightarrow A$ a free and proper action of a connected Lie group G by complete lifts. Suppose that we consider $\mu \in \mathfrak{g}^*$ such that $G = G_\mu$. Then, choosing any G -invariant section α_μ of A^* with values in $J_{A^*}^{-1}(\mu)$, there is a canonical Lie algebroid isomorphism*

$$\Upsilon_{\alpha_\mu} : ((\mathcal{T}^A A^*)_\mu, \Omega_\mu) \rightarrow (\mathcal{T}^{A_0} A_0^*, \Omega_{A_0} - (pr_1)^* B_\mu)$$

where A_0 is the vector bundle

$$A_0 = A/TG \rightarrow M/G$$

endowed with the Lie algebroid structure characterized by (5.34), Ω_{A_0} is the canonical symplectic-like structure on $\mathcal{T}^{A_0} A_0^*$, $pr_1 : \mathcal{T}^{A_0} A_0^* \rightarrow A_0$ is the projection on the first factor and $B_\mu \in \Gamma(\wedge^2 A_0^*)$ is the corresponding magnetic term associated with α_μ which is characterized by (5.43).

5.3. The general case. Let $(A, [\cdot, \cdot], \rho)$ be a Lie algebroid on the manifold M and $\Phi : G \times A \rightarrow A$ a free and proper action of a connected Lie group G on A by complete lifts with respect to the Lie algebra anti-morphism $\psi : \mathfrak{g} \rightarrow \Gamma(A)$.

Let $\mu \in \mathfrak{g}^*$ and denote by \mathfrak{g}_μ the isotropy algebra of μ . Then, the induced action $\Phi : G_\mu \times A \rightarrow A$ is a free and proper action by complete lifts with respect to the restriction $\psi : \mathfrak{g}_\mu \rightarrow \Gamma(A)$ of ψ to \mathfrak{g}_μ .

Now, denote by $\bar{\mu} \in \mathfrak{g}_\mu^*$ the restriction of μ to \mathfrak{g}_μ and by $J_{A^*}^\mu : A^* \rightarrow \mathfrak{g}_\mu^*$ the map given by

$$J_{A^*}^\mu = i^* \circ J_{A^*},$$

where $i^* : \mathfrak{g}^* \rightarrow \mathfrak{g}_\mu^*$ is the dual of the inclusion $i : \mathfrak{g}_\mu \rightarrow \mathfrak{g}$. Then, $J_{A^*}^\mu$ is the momentum map associated with the action of G_μ on A .

A direct computation proves that the isotropy group of $\bar{\mu} \in \mathfrak{g}_\mu$ with respect to the coadjoint action of G_μ , $(G_\mu)_{\bar{\mu}}$, is just G_μ . Therefore, we are in the conditions of Section 5.2 if we choose as the starting Lie group G_μ . Next, we choose a G_μ -invariant section $\alpha_\mu \in \Gamma(A^*)$ such that

$$\alpha_\mu(M) \subset (J_{A^*}^\mu)^{-1}(\bar{\mu}).$$

This is always possible as we have shown in Section 5.2. If $A_{0,\mu}$ is the vector bundle $A/TG_\mu \rightarrow M/G_\mu$ associated with the action $\Phi^T : TG_\mu \times A \rightarrow A$, we denote by $B_\mu \in \Gamma(\wedge^2 A_{0,\mu}^*)$ the corresponding magnetic term associated with α_μ . Then, from Theorem 5.2, we conclude that the reduced symplectic-like Lie algebroid

$$(\mathcal{T}^A A^*)_{\bar{\mu}} = (\mathcal{T}^A (J_{A^*}^\mu)^{-1}(\bar{\mu}))/TG_\mu \rightarrow J_{A^*}^{-1}(\mu)/G_\mu$$

is isomorphic to the symplectic-like Lie algebroid $(\mathcal{T}^{A_{0,\mu}} A_{0,\mu}^*, \Omega_{A_{0,\mu}} - pr_1^*(B_\mu))$, where $\Omega_{A_{0,\mu}}$ is the canonical symplectic-like structure on $\mathcal{T}^{A_{0,\mu}} A_{0,\mu}^*$ and $pr_1 : \mathcal{T}^{A_{0,\mu}} A_{0,\mu}^* \rightarrow A_{0,\mu}$ is the projection on the first factor.

On the other hand, the inclusion $i_{\mu,\bar{\mu}} : J_{A^*}^{-1}(\mu) \rightarrow (J_{A^*}^\mu)^{-1}(\bar{\mu})$ is G_μ -invariant and induces a Lie algebroid TG_μ -invariant monomorphism $I : \mathcal{T}^A J_{A^*}^{-1}(\mu) \rightarrow \mathcal{T}^A (J_{A^*}^\mu)^{-1}(\bar{\mu})$ over $i_{\mu,\bar{\mu}}$. Therefore, we have a Lie algebroid monomorphism $(\tilde{I}, \tilde{i}_{\mu,\bar{\mu}})$

$$\begin{array}{ccc} (\mathcal{T}^A J_{A^*}^{-1}(\mu))/TG_\mu & \xrightarrow{\tilde{I}} & (\mathcal{T}^A (J_{A^*}^\mu)^{-1}(\bar{\mu}))/TG_\mu \\ \tilde{\tau}_{\mathcal{T}^A J_{A^*}^{-1}(\mu)} \downarrow & & \downarrow \tilde{\tau}_{\mathcal{T}^A (J_{A^*}^\mu)^{-1}(\bar{\mu})} \\ J_{A^*}^{-1}(\mu)/G_\mu & \xrightarrow{\tilde{i}_{\mu,\bar{\mu}}} & (J_{A^*}^\mu)^{-1}(\bar{\mu})/G_\mu \end{array}$$

which is canonical with respect to Ω_μ and $\Omega_{\bar{\mu}}$ on the reduced spaces $(\mathcal{T}^A(J_{A^*})^{-1}(\bar{\mu}))/TG_\mu$ and $(\mathcal{T}^A(J_{A^*}^\mu)^{-1}(\bar{\mu}))/TG_\mu$, respectively.

Denote by $\tilde{\iota}_{\bar{\mu}} : \mathcal{T}^A(J_{A^*}^\mu)^{-1}(\bar{\mu}) \rightarrow \mathcal{T}^A A^*$ and by $\tilde{\iota}_\mu : \mathcal{T}^A(J_{A^*})^{-1}(\mu) \rightarrow \mathcal{T}^A A^*$ the corresponding inclusions which are related by

$$\tilde{\iota}_\mu = \tilde{\iota}_{\bar{\mu}} \circ I.$$

Now, if

$$\pi_{\bar{\mu}} : \mathcal{T}^A(J_{A^*}^\mu)^{-1}(\bar{\mu}) \rightarrow (\mathcal{T}^A(J_{A^*}^\mu)^{-1}(\bar{\mu}))/TG_\mu$$

$$\pi_\mu : \mathcal{T}^A(J_{A^*})^{-1}(\mu) \rightarrow (\mathcal{T}^A(J_{A^*})^{-1}(\mu))/TG_\mu$$

are the corresponding projections, we have that

$$\pi_\mu^* \Omega_\mu = \tilde{\iota}_\mu^* \Omega_A = I^* (\tilde{\iota}_{\bar{\mu}}^* \Omega_A) = I^* (\pi_{\bar{\mu}}^* \Omega_{\bar{\mu}}).$$

Then, using that $\pi_{\bar{\mu}} \circ I = \tilde{I} \circ \pi_\mu$ we conclude that

$$\tilde{I}^* \Omega_{\bar{\mu}} = \Omega_\mu.$$

Therefore, \tilde{I} is a canonical Lie algebroid monomorphism. Thus, we have proved the main result of this section.

Theorem 5.3. *Let $(A, [\cdot, \cdot], \rho)$ be a Lie algebroid over the manifold M and $\Phi : G \times A \rightarrow A$ a free and proper action of a connected Lie group by complete lifts. If $\mu \in \mathfrak{g}^*$ and $\bar{\mu}$ is the restriction of μ to \mathfrak{g}_μ , then, choosing a G_μ -invariant section α_μ of A^* with values in $(J_{A^*}^\mu)^{-1}(\bar{\mu})$, there exists a canonical embedding*

$$(\mathcal{T}^A A^*)_\mu \rightarrow \mathcal{T}^{A_{0,\mu}} A_{0,\mu}^*$$

from the reduced algebroid $(\mathcal{T}^A A^)_\mu$ equipped with the canonical reduced symplectic-like structure Ω_μ to the Lie algebroid $\mathcal{T}^{A_{0,\mu}} A_{0,\mu}^*$ endowed with the symplectic-like structure*

$$\bar{\Omega}_\mu = \Omega_{A_{0,\mu}} - (pr_1)^* B_\mu.$$

Moreover, this embedding is an isomorphism if and only if $\mathfrak{g} = \mathfrak{g}_\mu$.

Here $A_{0,\mu}$ is the vector bundle

$$A_{0,\mu} = A/TG_\mu \rightarrow M/G_\mu,$$

$\Omega_{A_{0,\mu}}$ is the canonical symplectic-like structure on $\mathcal{T}^{A_{0,\mu}} A_{0,\mu}^$, $pr_1 : \mathcal{T}^{A_{0,\mu}} A_{0,\mu}^* \rightarrow A_{0,\mu}$ is the projection on the first factor and $B_\mu \in \Gamma(\wedge^2 A_0^*)$ is the corresponding magnetic term associated with α_μ which is characterized by (5.43).*

Examples 5.4. (i) If we apply the previous theorem to the particular case when A is the standard Lie algebroid $TM \rightarrow M$ then we recover a classical result in cotangent bundle reduction theory (see [1, 12]).

(ii) For the case $A = \mathfrak{g} \times TM$ from (ii) in Examples 3.2, we have seen that the vector bundle $\mathcal{T}^A A^* \rightarrow A^*$ can be identified with $\mathfrak{g} \times T(\mathfrak{g}^* \times T^*M) \rightarrow \mathfrak{g}^* \times T^*M$ (see Example 4.1). Moreover, the Lie algebroid $(\mathfrak{g} \times TM)/TG \rightarrow M/G$ is isomorphic to the Atiyah algebroid associated with the principal bundle $\pi_M : M \rightarrow M/G$ (see (3.15)).

Then, the reduced symplectic-like Lie algebroid $(\mathcal{T}^A A^*)_0$, for the value $\mu = 0 \in \mathfrak{g}^*$, is symplectically isomorphic to the canonical cover of the fiberwise linear Poisson structure of $(T^*M)/G$

induced by the Atiyah Lie algebroid $(TM)/G \rightarrow M/G$, i.e. $\mathcal{T}^{(TM)/G}((T^*M)/G) \rightarrow (T^*M)/G$. In fact, this last Lie algebroid is just the Atiyah algebroid associated with the principal bundle $\pi_{T^*M} : T^*M \rightarrow (T^*M)/G$ (see [13]) and its symplectic-like structure $\Omega_{(T^*M)/G} \in \Gamma(\wedge^2(T^*(T^*M)/G))$ is the one induced by the G -invariant symplectic structure on T^*M .

Now, we choose $\mu \in \mathfrak{g}^*$ such that $G = G_\mu$ and a G -invariant 1-form $\alpha_\mu \in \Omega^1(M)$ on M such that $\alpha_\mu(M) \subset J^{-1}(\mu)$, where $J : T^*M \rightarrow \mathfrak{g}^*$ is the momentum map given as in (1.1). Then, the reduced symplectic-like Lie algebroid $(\mathcal{T}^A A^*)_\mu$ is simplyctically isomorphic to the Atiyah algebroid associated with the principal bundle $\pi_{T^*M} : T^*M \rightarrow (T^*M)/G$ endowed with the symplectic-like structure

$$\Omega_{(T^*M)/G} - \gamma_\mu$$

where $\gamma_\mu \in \Gamma(\wedge^2(T^*M/G))$ is the 2-section obtained from a magnetic term defined as follows. We consider the epimorphism

$$\tilde{\pi} : \mathfrak{g} \times TM \rightarrow TM/G, \quad \tilde{\pi}(\xi, v_x) = [v_x + \xi_M(x)].$$

Then, we have that there exists $B_\mu \in \Gamma(\wedge^2((T^*M)/G))$ such that

$$\tilde{\pi}^*(B_\mu) = (0, d\alpha_\mu).$$

Finally, γ_μ is just

$$(T\tau_{T^*Q}/G)^* B_\mu,$$

where $T\tau_{T^*Q}/G : (T(T^*M))/G \rightarrow (TM)/G$ is the vector bundle induced by the equivariant tangent lift $T\tau_{T^*M} : T(T^*M) \rightarrow TM$ of $\tau_{T^*M} : T^*M \rightarrow M$.

We finish this paper with an application which is related with the reduction of non-autonomous Hamiltonian systems.

Example 5.5. Let $p : M \rightarrow \mathbb{R}$ be a fibration. We denote by $\tau_{Vp} : Vp \rightarrow M$ the vertical bundle associated with p . Note that the sections of this vector bundle may be identified with the vector fields X on M such that $\eta(X) = 0$, where η is the exact 1-form $p^*(dt)$ on M , t being the standard coordinate on \mathbb{R} .

This vector bundle admits, in a natural way, a Lie algebroid structure where the Lie bracket is the standard Lie bracket of vector fields and the anchor map is the inclusion of vertical vectors with respect to p into TM .

Now, suppose that we additionally have a free and proper action $\phi : G \times M \rightarrow M$ of a Lie group G on M which is fibered, i.e.

$$p \circ \phi_g = p, \text{ for all } g \in G.$$

Then:

- (i) The infinitesimal generators of this last action are vertical vector fields.
- (ii) The tangent lifted action $T\phi : G \times TM \rightarrow TM$ induces a free and proper action

$$\Phi : G \times Vp \rightarrow Vp$$

of G on the vertical vector bundle Vp of p .

- (iii) p induces a new fibration $\tilde{p} : M/G \rightarrow \mathbb{R}$ on the quotient manifold M/G .

Then, $\Phi : G \times Vp \rightarrow Vp$ is an action by complete lifts with respect to the Lie algebra anti-morphism

$$\psi : \mathfrak{g} \rightarrow Vp, \quad \psi(\xi) = \xi_M.$$

Let μ be an element of \mathfrak{g}^* and we denote by $J_{V^*p} : V^*p \rightarrow \mathfrak{g}^*$ the momentum map defined as in (4.32). Then we have that the vector bundle $\mathcal{T}^{Vp}(J_{V^*p}^{-1}(\mu)) \rightarrow J_{V^*p}^{-1}(\mu)$ may be identified in a natural way with the vertical bundle $Vp_\mu \rightarrow J_{V^*p}^{-1}(\mu)$, where $p_\mu : J_{V^*p}^{-1}(\mu) \rightarrow \mathbb{R}$ is the fibration given by

$$p_\mu = p \circ \tau_{J_{V^*p}^{-1}(\mu)},$$

$\tau_{J_{V^*p}^{-1}(\mu)} : J_{V^*p}^{-1}(\mu) \rightarrow M$ being the corresponding projection. Under this identification the action $(\Phi, T\Phi^*) : G_\mu \times Vp_\mu \rightarrow Vp_\mu$ given by (4.28) is described as follows. Consider the tangent lift $T\Phi^* : G_\mu \times T(J_{V^*p}^{-1}(\mu)) \rightarrow T(J_{V^*p}^{-1}(\mu))$ of the restricted dual action $\Phi^* : G_\mu \times J_{V^*p}^{-1}(\mu) \rightarrow J_{V^*p}^{-1}(\mu)$. Since

$$p_\mu \circ \Phi^* = p_\mu,$$

we may induce an action of G_μ on Vp_μ which is just $(\Phi, T\Phi^*)$. Therefore, the action $(\Phi, T\Phi^*)^T : TG_\mu \times Vp_\mu \rightarrow Vp_\mu$ is given by

$$(\Phi, T\Phi^*)^T((g, \xi), v_{\alpha_x}) = T_{\alpha_x} \Phi_g^*(v_{\alpha_x} + \xi_M^*(\alpha_x)),$$

for all $(g, \xi) \in G_\mu \times \mathfrak{g}_\mu \cong TG_\mu$ and $\alpha_x \in V_x p$. Here $\xi_M^* \in \mathfrak{X}(V^*p)$ is the complete lift to V^*p of the infinitesimal generator of ξ with respect to the action ϕ .

Finally, from Theorem 3.11, we conclude that the reduced vector bundle

$$(Vp_\mu)/TG_\mu \rightarrow J_{V^*p}^{-1}(\mu)/G_\mu$$

is a symplectic-like Lie algebroid.

If $\mu = 0$, then using Theorem 5.1, we have that this symplectic-like Lie algebroid is isomorphic to $\mathcal{T}^{A_0} A_0^*$ where A_0 is the quotient vector bundle over M/G with total space Vp/TG . We remark that the action of $TG \cong G \times \mathfrak{g}$ on Vp is given by

$$\Phi^T((g, \xi), v_x) = T_x \phi_g(v_x + \xi_M(x)),$$

for $(g, \xi) \in G \times \mathfrak{g}$ and $v_x \in V_x p$. In fact, the vector bundle A_0 is isomorphic to the vertical bundle $V\tilde{p}$ with respect to the fibration $\tilde{p} : M/G \rightarrow \mathbb{R}$. The isomorphism is just

$$Vp/TG \rightarrow V\tilde{p}, \quad [v_x] \mapsto T_x \pi(v_x)$$

where $\pi : M \rightarrow M/G$ is the canonical projection. Therefore, $\mathcal{T}^{A_0} A_0^*$ may be identified with the vertical bundle $V\tilde{p} \rightarrow V^*\tilde{p}$ with

$$\tilde{p} = \tilde{p} \circ \tau_{V^*\tilde{p}} : V^*\tilde{p} \rightarrow \mathbb{R},$$

where $\tau_{V^*\tilde{p}} : V^*\tilde{p} \rightarrow M/G$ is the corresponding vector bundle projection. In conclusion, the reduced Lie algebroid $(Vp_0)/TG \rightarrow J_{V^*p}^{-1}(0)/G$ is canonically isomorphic to the Lie algebroid $V\tilde{p}$ on $V^*\tilde{p}$ with its standard symplectic-like structure.

Now, we consider $\mu \in \mathfrak{g}^*$ such that $G_\mu = G$. Let α_μ be a G -invariant 1-form on M such that

$$\alpha_\mu(\xi_M) = \mu(\xi), \quad \text{for all } \xi \in \mathfrak{g}.$$

Then, the restriction $\alpha_\mu|_{Vp}$ of α_μ to the vertical bundle Vp of the fibration $p : M \rightarrow \mathbb{R}$ determines a G -invariant section of V^*p with values in $J_{V^*p}^{-1}(\mu)$.

Let $\beta_\mu = d^{Vp}(\alpha_\mu|_{Vp})$. Equivalently, β_μ is the restriction of $d\alpha_\mu \in \Omega^2(M)$ to $Vp \times Vp$. The magnetic term B_μ associated with α_μ is the restriction to $V\tilde{p} \times V\tilde{p}$ of the unique 2-form \bar{B}_μ of M/G such that

$$\pi^* \bar{B}_\mu = d\alpha_\mu$$

where $\pi : M \rightarrow M/G$ is the quotient projection. Moreover, using Theorem 5.2, we have that $(Vp_\mu)/TG$ is a symplectic-like Lie algebroid on $J_{V^*p}^{-1}(\mu)/G$ isomorphic to the Lie algebroid $V\bar{p}$ endowed with the symplectic-like section

$$\Omega_{V\bar{p}} - pr_1^* B_\mu \in \Gamma(\wedge^2 V^* \bar{p})$$

with $pr_1 : V\bar{p} \rightarrow V\tilde{p}$ the vector bundle morphism on $\tau_{V^*\tilde{p}} : V^*\tilde{p} \rightarrow M/G$ given by the restriction to $V\bar{p}$ of the tangent lift $T\tau_{V^*\tilde{p}} : T(V^*\tilde{p}) \rightarrow T(M/G)$.

6. CONCLUSIONS AND FUTURE WORK

In this paper we have proved a reduction theorem for Lie algebroids with respect to a Lie group action by complete lifts. This result allows to obtain a Lie algebroid version of the classic Marsden-Weinstein reduction theorem for symplectic manifolds. We remark that as for the usual Marsden-Weinstein reduction theorem, the presence of the Lie group is superfluous, and the infinitesimal action of its Lie algebra is sufficient, although we have chosen not to use this approach here.

Additionally, in this paper the Marsden-Weinstein reduction process for symplectic-like Lie algebroids is applied to the particular case of the canonical cover of a fiberwise Poisson structure. It would be interesting to also obtain an analog version to the “bundle” or “fibrating” picture of cotangent bundle reduction in the setup of symplectic-like Lie algebroids, but this will be studied elsewhere.

It is also worth noticing that the classical Marsden-Weinstein reduction scheme does not only explain how to obtain a reduced symplectic structure on a quotient manifold, but it also shows that the reduced dynamics of a symmetric Hamiltonian function is again Hamiltonian with respect to this reduced symplectic structure. It is easy to prove that a similar phenomenon occurs for the reduction of symplectic-like algebroids by complete actions. In fact, under the same hypotheses as in Theorem 3.11, if $H : M \rightarrow \mathbb{R}$ is a G -invariant Hamiltonian function, then one can prove that the restriction to $J^{-1}(\mu)$ of H is a G_μ -invariant function, and thus, one can induce a real smooth function H_μ on $J^{-1}(\mu)/G_\mu$. Moreover, the restriction of the Hamiltonian section \mathcal{H}_H^Ω to $J^{-1}(\mu)$ is a section of the Lie algebroid $(J^T)^{-1}(0, \mu) \rightarrow J^{-1}(\mu)$ which is $(\tilde{\pi}_\mu, \pi_\mu)$ -projectable on the Hamiltonian section $\mathcal{H}_{H_\mu}^{\Omega_\mu}$. Thus, if $\gamma : I \rightarrow M$ is a solution of Hamilton's equations for H on the symplectic-like Lie algebroid $A \rightarrow M$ passing through a point in $J^{-1}(\mu)$, then the curve γ is contained in $J^{-1}(\mu)$ and $\pi_\mu \circ \gamma : I \rightarrow J^{-1}(\mu)/G_\mu$ is a solution of Hamilton's equations for H_μ on the reduced symplectic-like Lie algebroid $A_\mu \rightarrow J^{-1}(\mu)/G_\mu$.

In view of the results of this paper, one could apply this process to the reduction of symmetric Hamiltonian systems on Poisson manifolds. We have postponed this study for a future work.

REFERENCES

- [1] R. Abraham, J.E. Marsden: *Foundations of Mechanics*, (1987) second edition. Addison-Wesley Pub. Comp. Inc.
- [2] H. Bursztyn, G.R. Cavalcanti, M. Gualtieri, Marco: Reduction of Courant algebroids and generalized complex structures, *Adv. Math.* **211** (2007), no. 2, 726–765.
- [3] J.F. Cariñena, J.M. Nunes da Costa, P. Santos: Reduction of Lie algebroid structures, *Int. J. Geom. Methods Mod. Phys.* **2** (2005), 965–991.
- [4] T.J. Courant: Dirac manifolds, *Trans. A.M.S.*, **319** (1990), 631–661.
- [5] J. Grabowski, P. Urbański: Tangent and cotangent lifts and graded Lie algebras associated with Lie algebroids, *Ann. Global Anal. Geom.*, **15** (1997), 447–486.
- [6] J. Grabowski, P. Urbański: Lie algebroids and Poisson-Nijenhuis structures, *Rep. Math. Phys.*, **40** (1997), 195–208.
- [7] R.L. Fernandes: Lie algebroids, holonomy and characteristic classes, *Advances in Math.* **170** (2002) 119–179.
- [8] R. L. Fernandes, J.P. Ortega, T.S. Ratiu: The momentum map in Poisson geometry, *Amer. J. Math.* **131** (2009), no. 5, 1261–1310.
- [9] P.J. Higgins, K. Mackenzie: Algebraic constructions in the category of Lie algebroids, *J. Algebra*, **129** (1990), 194–230.
- [10] D. Iglesias, J.C. Marrero, D. Martín de Diego, E. Martínez, E. Padrón: Reduction of symplectic Lie algebroids by a Lie subalgebroid and a symmetry Lie group, *Symmetry, Integrability and Geometry: Methods and Applications* **3** (2007) 049, 28 pages.
- [11] Y. Kosmann-Schwarzbach: Exact Gerstenhaber algebras and Lie bialgebroids. Geometric and algebraic structures in differential equations, *Acta Appl. Math.* **41** (1995), no. 1-3, 153–165.
- [12] M. Kummer: On the construction of the reduced phase space of a Hamiltonian system with symmetry. *Indiana Univ. Math. J.* **30** 2, (1981) 281–291.
- [13] M. de León, J.C. Marrero, E. Martínez: Lagrangian submanifolds and dynamics on Lie algebroids, *J. Phys. A: Math. Gen.* **38** (2005) R241–R308.
- [14] Z.J. Liu, A. Weinstein, P. Xu: Manin triples for Lie bialgebroids, *J. Differential Geom.* **45** (1997), no. 3, 547–574.
- [15] K. Mackenzie: *General Theory of Lie Groupoids and Lie Algebroids*, London Mathematical Society Lecture Note Series: 213, Cambridge University Press, 2005.
- [16] K. Mackenzie, P. Xu: Lie bialgebroids and Poisson groupoids, *Duke Math. J.*, **73** (1994), 415–452.
- [17] K. Mackenzie, P. Xu: Classical lifting processes and multiplicative vector fields, *Quarterly J. Math.*, **49** (1998), 59–85.
- [18] J.E. Marsden, G. Misiolek, J.P. Ortega, M. Perlmutter, T.S. Ratiu: *Hamiltonian Reduction by Stages*, Lecture Note in Mathematics, Vol. 1913, Springer, 2007.
- [19] J.E. Marsden, T. Ratiu: Reduction of Poisson manifolds *Lett. Math. Phys.* V.11 (1986), 161–169.
- [20] J.E. Marsden, A. Weinstein: Reduction of symplectic manifolds with symmetry. *Rep. Mathematical Phys.* **5** 1 (1974) 121–130.
- [21] E. Martínez: Classical Field theory on Lie algebroids: multisymplectic formalism, arXiv:math/0411352.
- [22] E. Martínez: private communication.
- [23] J. P. Ortega, T.S. Ratiu: *Momentum maps and Hamiltonian reduction*, Progress in Mathematics, 222. Birkhäuser Boston, Inc., Boston, MA, 2004.
- [24] R.S. Palais: *A Global formulation of the Lie Theory of transformation groups*, Mem. AMS **22** (1957).
- [25] J.P. Roggiro Ayala: Lie algebroids over quotient spaces, *PhD Thesis*, Instituto Superior Tecnico, Lisbon, 2011.
- [26] J.W. Sater : Canonical reduction of mechanical systems invariant under abelian group actions with an application to celestial mechanics. *Indiana Univ. Math. J.* **26** (1977) 951–976.

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